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Budable Hypogeometric Polynomials: A Mathematical Study

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Abstract:

The theory of special functions and polynomial systems constitutes a fundamental area of mathematical analysis, with significant applications in differential equations, mathematical physics, approximation theory, and computational mathematics. Hypergeometric and hypogeometric series provide important analytical frameworks for studying functional representations, parameter-dependent identities, and finite polynomial expressions. The present study investigates the mathematical foundations relevant to the proposed topic of Budable Hypogeometric Polynomials, *with particular emphasis on series representations, parameter conditions, polynomial termination, and possible connections with differential equations. Since a standard definition of "Budable Hypogeometric Polynomials" could not be established from the references considered, the term is treated here as a provisional research designation rather than as a recognized polynomial family. Accordingly, the study examines the general mathematical principles that may support the development of a precise definition and a consistent theoretical framework for this proposed class of polynomials. Particular attention is given to the conditions under which a series terminates, the construction of finite algebraic expressions, and the criteria required to establish polynomial degree, coefficient structure, recurrence relations, and differential properties. The paper also outlines the mathematical investigations necessary to determine whether the proposed family possesses distinctive characteristics or meaningful relationships with existing special functions and classical orthogonal polynomials. The study thus provides a preliminary foundation for further research while maintaining a clear distinction between established mathematical results and properties that require independent verification.*

Keywords: Budable Hypogeometric Polynomials, Hypogeometric Series, Hypergeometric Functions, Polynomial Systems, Terminating Series, Differential Equations, Special Functions, Mathematical Analysis.

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Introduction:

Special functions and polynomial systems play a significant role in modern mathematical analysis because they provide systematic tools for representing functions, solving differential equations, and investigating a wide range of mathematical and physical phenomena. Their importance extends to mathematical physics, engineering, approximation theory, and computational methods. Among the major developments in this field, hypergeometric functions and related series occupy a prominent position owing to their well-defined analytical structures, parameter-dependent properties, and connections with numerous classical polynomial families.

A particularly important feature of hypergeometric-type series is that, under suitable parameter conditions, an infinite series may terminate after finitely many terms. When the resulting expression is a finite sum of non-negative integer powers of the independent variable, it defines a polynomial. Such terminating series provide a systematic approach to the construction and analysis of polynomial expressions. Their properties can be investigated through algebraic expansions, coefficient relations, recurrence formulas, differential equations, and, where applicable, orthogonality conditions.

The study of hypogeometric series and related mathematical structures requires careful attention to the definitions and conventions employed. The precise form of a series, the nature of its parameters, and the conditions governing convergence or termination determine its analytical and algebraic properties. Therefore, any attempt to develop a new polynomial system from such a framework must begin with an explicit definition, followed by a rigorous examination of its mathematical consistency and distinguishing characteristics.

The present paper focuses on *Budable Hypogeometric Polynomials* as the proposed subject of investigation. However, the term "Budable" could not be verified as part of a standard, established polynomial classification in the mathematical references considered. In the absence of an authoritative definition or a verified original source, it would be inappropriate to attribute specific properties, theorems, or applications to this term as though they were already established. The topic is therefore approached as a preliminary mathematical investigation intended to identify the definitions, assumptions, and analytical procedures necessary for developing a rigorous theory.

Within this framework, the study considers how finite polynomial expressions may arise from suitable series representations, how their coefficients and degrees can be characterized, and how their properties might be investigated using established methods from the theory of special functions. It also considers the criteria required to determine whether a proposed polynomial family is genuinely new or can be expressed in terms of existing mathematical objects. Such comparisons are essential for establishing mathematical originality and avoiding unnecessary duplication of known results.

The principal aim of this paper is consequently to establish a sound conceptual and analytical basis for the further study of Budable Hypogeometric Polynomials. Rather than presupposing an unverified mathematical structure, the paper identifies the questions that must be resolved before a complete theory can be formulated. This approach supports the development of a precise definition, verifiable

propositions, and potential applications grounded in rigorous mathematical reasoning.

Objectives of the Study:

The principal objective of this study is to formulate a mathematically rigorous starting point for investigating the proposed class of Budable Hypogeometric Polynomials. The specific objectives are as follows:

1. **To examine the theoretical foundations** of hypogeometric series and related special functions, with particular attention to their definitions, parameter conditions, and algebraic structures.
2. **To investigate the formation of polynomial expressions** from terminating series and identify the conditions under which a finite series defines a polynomial of a specified degree.
3. **To analyze the role of parameters and coefficients** in determining the degree, algebraic form, and structural properties of the resulting polynomial expressions.
4. **To establish the requirements for a precise definition** of Budable Hypogeometric Polynomials, including their domain, parameter restrictions, defining relations, and conditions of mathematical validity.
5. **To explore possible connections with differential equations and classical polynomial systems**, while distinguishing verified relationships from conjectural extensions.
6. **To identify appropriate analytical methods** for investigating recurrence relations, differential properties, zeros, orthogonality, and other characteristics, where these properties can be rigorously established.
7. **To assess the mathematical originality of the proposed family** by comparing its eventual definition with existing hypergeometric functions, hypogeometric series, and established polynomial systems.
8. **To identify directions for further research** that may lead to a consistent theoretical framework and, where justified by mathematical evidence, potential applications in analysis and related disciplines.

Research Methodology:

The present study adopts a theoretical, analytical, and literature-based research methodology to investigate the mathematical foundations relevant to the proposed topic of *Budable Hypogeometric Polynomials*. The approach is primarily deductive, relying on established mathematical definitions, algebraic reasoning, symbolic calculations, and the critical examination of relationships among special-function series and polynomial systems. The principal purpose of this methodology is to develop a rigorous conceptual framework within which the proposed polynomial family can be defined, analyzed, and evaluated mathematically.

The investigation begins with a review of relevant literature on hypergeometric functions, hypogeometric series, terminating series, and classical polynomial systems. This review is intended to establish the existing mathematical framework, clarify the terminology used in the study, and identify established results that may be relevant to the proposed research. Particular attention is given to the definitions of the series under consideration, the role of their parameters, and the conditions governing convergence,

termination, and the formation of finite algebraic expressions.

The analytical component of the research examines the relationship between parameter selection and the termination of series expansions. Where the relevant definitions permit, finite-series expressions are constructed and their algebraic forms are examined to determine whether they satisfy the mathematical criteria for polynomial representations. The resulting expressions may then be analyzed with respect to their degrees, coefficients, and other properties that follow directly from their defining formulas. All calculations are intended to be supported by explicit mathematical reasoning so that the results can be independently checked and reproduced.

The methodology also includes a comparative examination of the proposed mathematical structure and existing special functions and polynomial families. Such a comparison is necessary to determine whether the proposed expressions represent a distinct mathematical construction or can be reduced to, or interpreted through, already established functions and polynomials. Depending on the eventual definition, relevant properties may include recurrence relations, differential equations, generating functions, zeros, and orthogonality. These properties will be investigated only where sufficient mathematical definitions and supporting derivations are available.

The research proceeds through the following stages:

1. **Literature Review:** Examine established definitions, classifications, and fundamental results concerning hypergeometric functions, hypogeometric series, and related polynomial systems.
2. **Definition and Terminology:** Clarify the mathematical meaning of the terminology used in the study and identify the precise definition required for Budable Hypogeometric Polynomials.
3. **Parameter Analysis:** Investigate the influence of parameter choices on series termination, coefficient formation, and the degree of finite polynomial expressions.
4. **Algebraic Construction:** Develop illustrative finite-series expressions where justified by the adopted definitions and verify their polynomial character through direct algebraic analysis.
5. **Comparative Analysis:** Compare the resulting expressions with established special functions and polynomial families to identify possible equivalences, extensions, or distinctive features.
6. **Verification of Results:** Check calculations and proposed identities through explicit derivations, consistency checks, and, where appropriate, symbolic computation. Any general proposition will require a mathematical proof rather than reliance on numerical examples alone.
7. **Identification of Research Directions:** Determine which properties can be established from the available definitions and identify questions requiring further investigation.

An important methodological consideration is that the term *Budable Hypogeometric Polynomials* has not yet been assigned a verified standard definition in the references considered for this study. Consequently, the mathematical examples presented at this stage serve to explain the general principles underlying terminating series

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and polynomial construction; they do not establish properties specific to a formally defined Budable polynomial family. A definitive investigation of such a family will require an explicit defining formula, appropriate parameter restrictions, and a systematic derivation of its properties.

Through this methodology, the study seeks to maintain mathematical precision, reproducibility, and a clear distinction between established results, illustrative calculations, and hypotheses requiring further verification. This provides a systematic foundation for developing a more complete theory of Budable Hypogeometric Polynomials once their defining mathematical structure has been established.

Mathematical Preliminaries:

The study of special functions and polynomial systems requires a clear understanding of the fundamental mathematical concepts, notations, and analytical techniques used to represent and investigate these functions. In particular, the theory of series expansions provides an important framework for understanding the relationship between infinite series, finite algebraic expressions, and polynomial solutions of differential equations. The rising factorial, the gamma function, and parameter-dependent series are among the basic mathematical tools employed in this area. These concepts are particularly relevant when investigating whether a given series can be transformed into a finite polynomial expression and whether the resulting expression possesses identifiable algebraic or analytical properties.

The present section introduces the mathematical preliminaries required for the proposed study of *Budable Hypogeometric Polynomials*. It begins with the definition and principal properties of the rising factorial, followed by a discussion of a parameter-dependent series representation, its coefficient structure, and its convergence conditions. The section also explains the distinction between an infinite special-function series and a terminating polynomial expression. These preliminaries establish a general mathematical foundation for further investigation without assuming that the proposed class of Budable Hypogeometric Polynomials has already been formally defined.

Rising Factorial and Pochhammer Notation:

The rising factorial, also known as the Pochhammer symbol, is widely used in the study of special functions and hypergeometric series. It provides a compact representation of products of consecutive factors.

For a parameter a and a non-negative integer n, it is defined as

$$(a)_0=1$$

$$(a)_n=a(a+1)(a+2)\cdots(a+n-1)=\prod_{k=0}^{n-1}(a+k),\quad n\geq1.$$

The first few values are

$$\begin{aligned} (a)_1&=a,\\ (a)_2&=a(a+1),\\ (a)_3&=a(a+1)(a+2). \end{aligned}$$

The rising factorial is also related to the gamma function by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},$$

whenever the quotient is defined. Its important recurrence relations are

$$(a)_{n+1}=(a+n)(a)_n$$

and

$$(a)_{m+n}=(a)_m(a+m)_n.$$

These identities simplify algebraic calculations and the representation of series coefficients. In the present research, Pochhammer notation provides a useful foundation for studying parameter-dependent series and the conditions under which such series may define polynomials.[1]

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A Parameter-Dependent Hypogeometric-Type Series:

A parameter-dependent series is an important tool in the study of special functions. Its coefficients depend on one or more parameters, and its properties are determined by the coefficient structure, parameter restrictions, and convergence behavior.

For preliminary illustration, consider the series

$$\boxed{F(;c;z)=\sum_{n=0}^{\infty}\frac{z^n}{n!(c)_n}}$$

where z is the independent variable, c is a parameter, and $(c)_n$ denotes the rising factorial. Expanding the series gives

$$\begin{aligned} F(;c;z) ={}&1+\frac{z}{c} +\frac{z^2}{2c(c+1)}\\ &+\frac{z^3}{6c(c+1)(c+2)}\\ &+\frac{z^4}{24c(c+1)(c+2)(c+3)} +\cdots. \end{aligned}$$

The general coefficient of z^n is

$$\boxed{A_n=\frac{1}{n!(c)_n}}.$$

Using the identities $(n+1)!=(n+1)n!$ and $(c)_{n+1}=(c+n)(c)_n$, we obtain the recurrence relation

$$\boxed{A_{n+1}=\frac{A_n}{(n+1)(c+n)}}.$$

This relation simplifies the calculation of successive coefficients and provides a useful basis for studying convergence and parameter dependence. For the displayed series, the parameter must satisfy $c\notin\{0,-1,-2,\ldots\}$. The ratio test shows that the series converges for every finite z.

The notation $F(;c;z)$ is used here only to represent this particular series. Its identification with a standard hypogeometric function, and its relevance to Budable Hypogeometric Polynomials, require confirmation from an authoritative definition or source.

Parameter Restrictions and Convergence:

The validity of a series depends on the permissible values of its parameters. Since the denominator contains the rising factorial $(c)_n$, the parameter must satisfy

$$\boxed{c\notin\{0,-1,-2,-3,\ldots\}}.$$

The convergence of the series can be examined using the ratio test. For $z\neq0$, the ratio of consecutive terms is

$$R_n=\left|\frac{z}{(n+1)(c+n)}\right|.$$

Taking the limit gives

$$\lim_{n\to\infty}R_n=0.$$

Since this limit is less than one for every finite z, the series converges absolutely for all finite complex values of z, provided the parameter restriction is satisfied.

Finite Polynomial Expressions:

A polynomial of degree at most N can be expressed as

$$\boxed{P_N(x)=\sum_{k=0}^{N}a_kx^k},$$

where $a_0,a_1,\ldots,a_N$ are its coefficients. A special-function series defines a polynomial when it terminates after finitely many terms. In generalized hypergeometric series, termination can occur when an upper parameter is a non-positive integer. However, this property cannot be assumed for every series without examining its definition.

Therefore, the study of Budable Hypogeometric Polynomials requires a precise definition of the series, its parameters, and the conditions under which it terminates.

The Proposed Concept of Budable Hypogeometric Polynomials:

The mathematical definition of "Budable Hypogeometric Polynomials" must be established before their properties can be investigated. In particular, their coefficients, degree, generating function, and differential

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equations depend on the precise definition of the family.

A general polynomial family may be represented as

$$\boxed{P_n(x)=\sum_{k=0}^{n}c_{n,k}(\boldsymbol{\lambda})x^k},$$

where n is a non-negative integer, $\boldsymbol{\lambda}$ represents the parameters, and $c_{n,k}$ denotes the coefficient of x^k.

This expression provides a general framework for polynomial families; it is not a verified definition of Budable Hypogeometric Polynomials. An authoritative source or the original defining formula is needed to develop their specific mathematical properties.

Mathematical Definition of the Proposed Polynomial Family:

A rigorous definition of a polynomial family requires an explicit rule for determining its coefficients. These coefficients may be specified through a finite series, a recurrence relation, a generating function, or a differential equation. The chosen definition determines the algebraic and analytical properties of the resulting polynomials.

For the proposed Budable Hypogeometric Polynomials, the defining formula must first be established. Until then, their specific properties cannot be determined conclusively.

Conditions for a Rigorous Polynomial Definition:

A complete mathematical definition should specify the parameter domain, coefficient formula, polynomial degree, and relationship with any known special functions. It must also establish each claimed property through mathematical proof. Properties such as orthogonality, real-rootedness, positivity, and differential equations cannot be assumed without verification.

These requirements provide a systematic foundation for investigating a new family of special-function polynomials.

Illustrative Mathematical Construction:

Consider the following polynomial expressions:

$$P_0(x)=1,$$

$$P_1(x)=a_0+a_1x,$$

$$P_2(x)=b_0+b_1x+b_2x^2.$$

The coefficients may depend on specified parameters. For $P_2(x)$ to have degree exactly two, the leading coefficient must satisfy

$$b_2\neq0.$$

More generally, consider

$$\boxed{P_n(x)=\sum_{k=0}^{n}a_{n,k}x^k}.$$

The polynomial has degree exactly n if and only if

$$\boxed{a_{n,n}\neq0}.$$

This condition is essential when determining the degree and leading coefficient of a polynomial family. However, the coefficients must be defined explicitly before further properties can be established. The expressions above are illustrative examples, not a verified definition of Budable Hypogeometric Polynomials.

Differential Equations and Polynomial Families:

Differential equations are widely used to characterize special functions and polynomial families. A general second-order linear differential equation may be written as

$$A(x)y''+B(x)y'+\lambda_n y=0,$$

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where \(A(x)\) and \(B(x)\) are specified functions and \(\lambda_n\) is a parameter associated with the polynomial degree.

For a proposed polynomial family \(P_n(x)\), the investigation may examine whether its members satisfy such an equation, which parameter restrictions are required, and whether polynomial solutions exist for every non-negative integer \(n\). It may also explore the uniqueness of solutions and the derivation of recurrence relations.

These questions can form the basis of further research into Budable Hypogeometric Polynomials. However, their answers require a verified defining formula or an authoritative mathematical source.

Recurrence Relations and Generating Functions:

Recurrence relations and generating functions are fundamental tools in the study of special functions and polynomial families. They provide systematic methods for constructing polynomial sequences, investigating their algebraic structures, and deriving relationships between polynomials of different degrees. These methods are particularly useful when a polynomial family is defined through parameter-dependent coefficients or finite series.

A recurrence relation expresses a polynomial in terms of one or more polynomials of lower degree. In many classical polynomial systems, a three-term recurrence relation takes the general form

\[\boxed{ P_{n+1}(x)=(A_nx+B_n)P_n(x)-C_nP_{n-1}(x) } \]

where \(P_n(x)\) denotes the polynomial of degree \(n\), while \(A_n\), \(B_n\), and \(C_n\) are coefficients determined by the particular family. The initial polynomials and the values of these coefficients must be specified to generate the sequence uniquely.

The importance of a recurrence relation lies in its ability to reduce the calculation of higher-degree polynomials to previously obtained expressions. For example, once \(P_0(x)\) and \(P_1(x)\) are known, a valid three-term recurrence may be used to construct \(P_2(x)\), \(P_3(x)\), and subsequent members of the family. Recurrence relations may also help establish identities involving leading coefficients, polynomial degrees, and parameter dependence. However, the existence of a three-term recurrence is not guaranteed for every polynomial family and must be demonstrated mathematically.

A generating function provides another systematic approach to polynomial analysis. It represents an entire sequence of polynomials through a single function of two variables. A commonly used formal representation is

\[\boxed{G(x,t)=\sum_{n=0}^{\infty}P_n(x)t^n } \]

where \(x\) is the polynomial variable and \(t\) is an auxiliary variable. The generating function can encode information about all members of the sequence and may simplify the derivation of recurrence relations, coefficient identities, and differential equations. Depending on the family, it may also provide a convenient method for studying parameter dependence and obtaining explicit expressions for individual polynomials.

For the proposed Budable Hypogeometric Polynomials, recurrence relations and generating functions could form important areas of investigation. The first step, however, is to establish a precise definition of the family and determine its coefficients. Once this definition is available, it may be possible to investigate whether a recurrence relation exists,

whether a generating function can be derived, and what mathematical properties follow from these representations. Until such results are proved, these relations should be regarded as research objectives rather than established properties of the proposed polynomials.

Orthogonality and the Distribution of Zeros:

Orthogonality is an important concept in the theory of classical orthogonal polynomials and has significant applications in approximation theory, numerical analysis, mathematical physics, and differential equations. It describes a relationship between different members of a polynomial family through integration against a specified weight function.

A polynomial family $\{P_n(x)\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a,b]$ if

$$\boxed{ \int_a^b P_m(x)P_n(x)w(x)\,dx=0, \qquad m\neq n. }$$

Here, $w(x)$ is the weight function, and the interval and parameter restrictions must be chosen so that the relevant integrals exist. In the usual real-valued setting, the weight function is non-negative and integrable, with additional conditions imposed to ensure that the associated inner product is well-defined.

Orthogonality allows functions to be represented or approximated using expansions in terms of suitable polynomial sequences. It is especially valuable in numerical integration, where orthogonal polynomials can be used to construct quadrature rules, and in approximation theory, where they provide useful bases for representing functions. In mathematical physics, such polynomials also arise in the solutions of certain differential equations and boundary-value problems.

For a proposed polynomial family, establishing orthogonality requires more than verifying that its members are polynomials. One must identify an appropriate weight function, specify the interval of integration, establish the admissible parameter values, and prove that the integral vanishes for distinct polynomial degrees. If these conditions cannot be satisfied, the family may still be mathematically useful, but it should not be classified as an orthogonal polynomial system without further justification.

The distribution of zeros is another significant aspect of polynomial theory. The zeros of a polynomial are the values of x for which $P_n(x)=0$. Their locations, multiplicities, and relationships across different degrees can reveal important structural properties of a polynomial family. For example, research may examine whether the zeros are real, whether they are distinct, whether they lie within a particular interval, and how their positions change as the degree or parameters vary.

For classical orthogonal polynomial systems, additional results concerning the reality and location of zeros can often be established under suitable assumptions on the weight function and interval. Such results are important in numerical methods and approximation theory because the positions of polynomial zeros can influence the stability and accuracy of computational procedures. Nevertheless, these conclusions cannot automatically be extended to an unfamiliar or newly proposed polynomial family.

In the case of Budable Hypogeometric Polynomials, orthogonality and zero distribution should therefore be treated as questions for further investigation. After the defining formula has been verified, one may examine the roots of the first few polynomials,

investigate possible patterns, and formulate conjectures for subsequent mathematical proof. Any conclusions concerning real-rootedness, simplicity of zeros, or orthogonality must be supported by rigorous arguments rather than numerical observations alone.

In summary, recurrence relations, generating functions, orthogonality, and zero distribution provide a useful framework for studying the structure and potential applications of special-function polynomial families. Their investigation may contribute to a systematic mathematical theory of Budable Hypogeometric Polynomials, provided that the family is first defined precisely and its fundamental properties are established.

Potential Applications: The possible applications of Budable Hypogeometric Polynomials depend on the precise mathematical structure eventually assigned to them.

If the family admits a useful differential equation, it may be relevant to the study of boundary-value problems. If it has a suitable orthogonality relation, it may be considered for polynomial approximation or numerical integration. If it possesses an efficient recurrence relation, it may be useful in symbolic or numerical computation.

Similarly, a generating function or transformation identity could connect the family with established special functions.

At the present stage, these are potential applications rather than demonstrated results. Establishing practical value would require explicit formulas, mathematical proofs, and, where appropriate, numerical comparisons with existing methods.

Future Scope of Research:

Further research on the proposed topic should begin with verification of its terminology and definition. The following stages would provide a systematic direction:

1. Locate the original source in which the term "Budable Hypogeometric Polynomials" appears.
2. Establish the precise defining series or recurrence relation.
3. Determine the parameter domain and prove the polynomial degree.
4. Derive initial polynomial members and verify their coefficients.
5. Investigate possible differential equations and generating functions.
6. Examine orthogonality and the distribution of zeros, if the defining structure permits.
7. Compare the resulting family with known special functions and polynomial systems.
8. Identify a specific result that extends the existing literature.

The last stage is particularly important for doctoral research. A PhD-level contribution should go beyond reproducing known definitions and formulas by developing a well-defined problem and proving a new result or producing a justified new application.

Conclusion:

The present study has outlined the mathematical foundations necessary for investigating the proposed topic of Budable Hypogeometric Polynomials. Particular attention has been given to rising factorial notation, parameter-dependent series, convergence conditions, polynomial termination, recurrence relations, generating functions, differential equations, orthogonality, and the distribution of zeros. These concepts

provide a general framework for understanding the structure and properties of special-function polynomial families.

The discussion demonstrates that a rigorous mathematical investigation must begin with a precise definition of the proposed polynomial family. The defining expression should specify the coefficient structure, admissible parameter values, and conditions under which the series terminates to produce a polynomial. Once these fundamental aspects are established, further analysis can be undertaken to determine the degree, leading coefficients, recurrence relations, generating functions, and possible differential equations associated with the family.

The investigation of orthogonality and zero distribution may also provide valuable directions for future research. However, such properties must be established through appropriate mathematical proofs rather than assumed from the terminology or general structure of a special-function series. Similarly, potential applications in approximation theory, numerical analysis, and mathematical physics can be considered only after the relevant properties have been demonstrated.

An important limitation of the present study is that the term Budable Hypogeometric Polynomials has not yet been verified through an authoritative mathematical definition. Therefore, the general formulas and analytical methods discussed in this paper should be regarded as preliminary mathematical background rather than established results concerning this specific family.

In conclusion, establishing the precise definition of Budable Hypogeometric Polynomials is the essential first step toward developing a complete mathematical theory. Once the original defining formula or an authoritative reference has been identified, its properties can be investigated systematically and compared with existing classes of special functions and orthogonal polynomials. This approach will provide a sound foundation for determining the mathematical significance, originality, and potential applications of the proposed polynomial family.

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