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PRINCIPALS POLYNOMIAL INTEGRALS OF QUADRATIC SYSTEMS

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Abstract:

The primary objectives of this study are to (1) provide a classification for all quadratic systems with minimal polynomial first integrals of degree less than 5 and (2) demonstrate the existence of minimal polynomial first integrals for quadratic systems of any degree. The topological phase portraits of these systems will also be presented. In addition to this, we show that quadratic systems with minimum polynomial first integrals of degree more than one have at most three invariant straight lines, and given certain reasonable assumptions, we offer the largest degree of the irreducible polynomial firstintegrals.

Keywords: *Polynomial differential system, minimal polynomial first integral, phase portrait.*

Introduction:

A polynomial system is a differential system that must have the form according to its specification.

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y),$$

where the dependent variables x and y, as well as the independent variable (the time), t, are all real, and P, Q R[x, y], where, as is customary, R[x, y] indicates the ring of polynomials in the variables x and y with real coefficients. In what follows, every function that is described is defined in R[x, y], and every constant is a real value. It may be said that the degree of the polynomial system is equal to the maximum value of the degrees P and Q. In the following, we shall refer to polynomial systems of degree 2 as quadratic systems.

Quadratic systems have been the subject of much research, and as a result, almost one thousand academic articles have been written regarding these systems. However, there is no clear answer to the question of which quadratic systems can be integrated. Because the definition of what it means for a polynomial system to be integrable varies depending on the author (for example, see [8]), we need to specify what we mean by the term.

A traditional method for classifying all of the possible trajectories of a polynomial system is the search for first integrals of the system. Darboux [10] demonstrated in 1878 how the first integrals of planar polynomial systems that have adequate invariant algebraic curves could be built. His proof was based on the fact that these systems may contain invariant curves. Jouanolou [13] in 1979, Prelle and Singer in 1983, and Singer in 1992 are mostly responsible for the most significant advancements that have been made to Darboux's findings about planar polynomial systems. Recent years have seen a number of scholars produce some very fascinating findings in connection with Darboux's notion of integrability (see, for instance, Kooij and Christopher [14], Zholadek, Chavarriga, Giacomini, Gine and Llibre [8], etc.). In point of fact, the Darboux theory of integrability applies to polynomial differential systems in arbitrarily finite dimension; for example, Christopher and Llibre [9] describe how this might be the case.

This study looks at whether or not polynomial first integrals exist for quadratic systems (1) and provides the topological phase pictures that correspond to such integrals. Moulin-Ollagnier [19] and Labrunie [16] have described the polynomial first integrals of a unique three-dimensional Lotka-Volterra system, the so-called abc system, which may be written as follows: I Moulin-Ollagnier [19] and [16]

$$\dot{x} = x(cy + z), \qquad \dot{y} = y(x + az), \qquad \dot{z} = z(bx +$$

The polynomial first integrals of any degree for the two-dimensional Lotka-Volterra quadratic system were given by Cair'o and Llibre [7], and the form was as follows:

$$\dot{x} = x(a_1 + b_{11}x + b_{12}y), \qquad \dot{y} = y(a_2 + b_{21}x + b_{22}y).$$

The following is the structure of the paper: In Section 2, we provide several essential definitions that will come in handy in subsequent sections. In Section 3, we lay out our seven key theorems, among which Theorems A, B, C, and F describe all topological phase pictures of quadratic systems with minimum polynomial first integrals of degree 1, 2, 3, and 4, respectively. These theorems may be found in the previous section. The categorization of quadratic systems based on whether or not they have a minimum polynomial first integral of degree 4 is provided by theorems D and E. The proof that quadratic systems with more than three invariant straight lines can't have minimum polynomial first integrals of degree more than one may be found in the theorem known as G. We show in Section 4 that there are quadratic systems with minimal polynomial first integrals of any degree, and we show that if a quadratic system has a polynomial first integral H(x, y) such that H(x, y) + c is irreducible in R[x, y] for any x and y, then there are quadratic systems with minimal polynomial first integrals of any degree.

Topological Equivalence:

In the remaining sections of this article, we will say that polynomial vector fields X and Y on R2 are topologically equivalent if there exists a homeomorphism on S2 that preserves the circle at infinity S1 and carries orbits of the flow induced by p(X) into orbits of the flow induced by p(Y), while simultaneously preserving or reversing the sense of all orbits. Although this notion of topological equivalence is not the standard one, it is one that makes the study of phase pictures of polynomial vector fields relatively straightforward.

A separatrix of p(X) might be an orbit that is a single point, a limit cycle, or a trajectory that sits in the boundary of a hyperbolic sector at a singular point. Each of these three possibilities is an example of an orbit that is a separatrix. If the initial integral of a quadratic system is polynomial, then the system does not have any limit cycles. This is a straightforward conclusion to reach when one considers the fact that a polynomial first integral is a continuous function that is defined throughout the whole plane. We refer to the set that is created by all separatrices of p using the notation S(p(X)) (X). Neumann [20] presented evidence to demonstrate that the set S(p(X)) is, in fact, closed. The term "canonical region of p" refers to any component of S2 S(p(X)) that is open and linked (X). One way to define a separatrix configuration is as the union of S(p(X)) with one representative solution selected from each canonical area. If there is a homeomorphism on S2 that can transport orbits of S(p(X)) into orbits of S(p(Y)), while simultaneously retaining or reversing the meaning of all orbits, then we say that S(p(X)) and S(p(Y) are equivalent to one another. The following theorem, attributed to Neumann [20], specifies the characterisation of two Poincar'ecompactified vector fields that are topologically equal to one another. In the future, when we conduct the analysis of the global phase portraits of the quadratic system (1) including a polynomial initial integral, we will need its use.

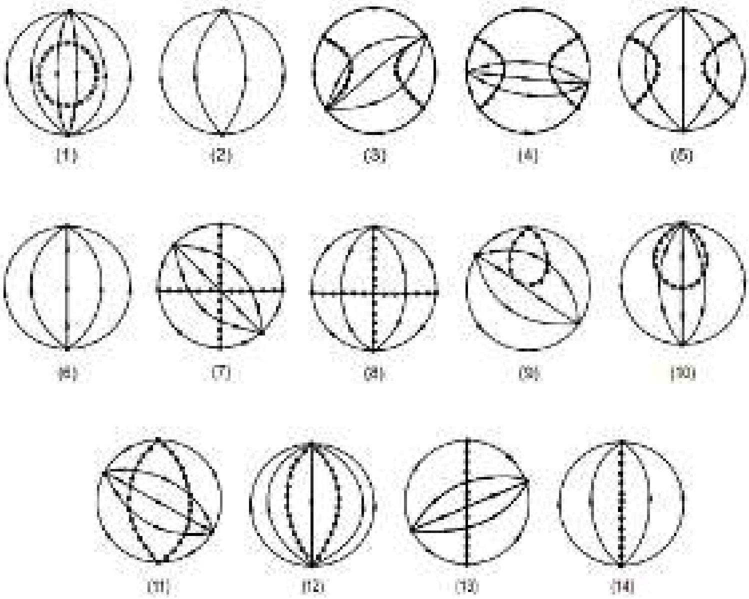

Figure 1. Phase portraits of quadratic systems having a polynomial first integral of degree 1.

Where the sum of a2 and b2 doesn't equal 0. In addition to this, each of the related phase pictures is topologically equal to one of the eight different phase portraits shown in Figure 2. Every phase picture in Figure 2 may be achieved by some X2 I where I falls within the range [1, 2, 3].

We make the observation that any quadratic system with a minimum polynomial first integral of degree 2 has a single straight line that is constructed by singular points in a singular manner. Our third discovery describes the classification as well as the topological phase pictures of quadratic systems (1) that have a minimum polynomial initial integral of degree 3.

Outside of the line A = 0, the quadraticvector field (P, Q) is equivalent, which scales the variable t to a linear system. According to Proposition 3(b) of [4,] it may have a maximum of two invariant straight lines. Given this, the conclusion is obvious. For the time being, let us suppose that P and Q both have relatively prime values. It is well knowledge that a quadratic system (1) has a maximum of five straight lines that do not change (for example, see the corollary in [4] or [26] titled "Corollary 5(a)"). Because the evidence for Proposition 1 demonstrates that deg H is less than 3, we know that. As a result of the demonstrations of Theorems C and F, we are aware that quadratic systems with a minimum polynomial first integral of degree 3 and 4 have no more than three invariant straight lines at most. In the following, we shall demonstrate, with the assistance of Lemma 3, that quadratic systems with more than three invariant straight lines do not contain any polynomial first integrals with degrees more than 4. Since only systems (II), (VII), and (VIII) have a unique vertical invariant straight line, system (V) has two unique vertical invariant straight lines, and system (X) has an infinite number of invariant straight lines, we only need to study the invariant straight lines of the form in order to avoid losing any generality. System (V) has two unique vertical invariant straight lines.

$$y = kx + c, \tag{24}$$

for the first nine systems of Lemma 3. In what follows we prove our results for the nine systems one by one.

For system (I) the straight line (24) is invariant if and only if the following condition holds

$$Q(x, kx + c) \equiv k[1 + x(kx + c)].$$

This condition is verified if and only if the following three conditions hold:

$$\begin{aligned} k &= b_{00} + b_{01}c + b_{02}c^2, \\ b_{10} + b_{11}c &+ (b_{01} - c + 2b_{02}c)k = 0, \\ b_{20} + b_{11}k &+ (b_{02} - 1)k^2 = 0. \end{aligned} \tag{25}$$

Substituting k from the first equation of (25) into the second and third equations of (25) yields

(26)

$$\begin{aligned} & b_{00}b_{01} + b_{10} + (-b_{00} + b_{01}^2 + 2b_{00}b_{02} + b_{11})c \\ & \quad + (-b_{01} + 3b_{01}b_{02})c^2 + (-b_{02} + 2b_{02}^2)c^3 = 0, \\ & -b_{00}^2 + b_{00}^2 b_{02} + b_{00}b_{11} + b_{20} + (-2b_{00}b_{01} + 2b_{00}b_{01}b_{02} + b_{01}b_{11})c \\ & \quad + (-b_{01}^2 - 2b_{00}b_{02} + b_{01}^2 b_{02} + 2b_{00}b_{02}^2 + b_{11}b_{02})c^2 \\ & \quad + (-2b_{01}b_{02} + 2b_{01}b_{02}^2)c^3 + (-b_{02}^2 + b_{02}^3)c^4 = 0. \end{aligned}$$

For system (I) the straight line (24) is invariant if and only if the following condition holds

$$Q(x, kx + c) \equiv k[1 + x(kx + c)].$$

Conclusion:

Where the stars stand for the components that almost certainly have a nonzero value. It may be shown, in a manner similar to the proof of system (I), that system (V) does not include any polynomial first integrals with degrees greater than 4. We are able to demonstrate the following assertions about system (VI) (IX) by using the same strategy that we used when discussing system (V), which is how we can do the following proofs: Systems (VI) and (IX) have no polynomial first integrals with degrees more than 4 if b02 = 0 and have a maximum of one invariant straight line if b02 = 0. Additionally, if b02 = 0 there is only one invariant straight line if b02 = 0. Systems (VII) and (VIII) contain no polynomial first integrals of degree more than 4 if b02

= 0 and a maximum of two invariant straight lines if b02 = 0. Additionally, if b02 = 0 there are no more than two invariant lines if b02 = 0. The conclusion that follows from the previous findings is known as Theorem G.

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