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An Investigation Into A Number Of Different Types Of Analytic Functions Involving The Jackson Q-Difference Operator

Asma Fatima Abdul Jabbar[1] & Dr.Vineeta Basotia[2]

[1]*Ph.D. Research Scholar, Department of Mathematics, Shri. J.J.T. University, Rajasthan, India*

[2]*Professor and Research Guide, Department of Mathematics, Shri. J.J.T. University, Rajasthan, India*

Corresponding Author - Asma Fatima Abdul Jabbar

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Abstract:

At the turn of the previous century, Frank Hilton Jackson was responsible for the methodical development of q-calculus as a direct consequence of the groundbreaking work done by Euler and Heine. Jackson is responsible for developing the principles of the q-derivative via the work that he did. By making use of a linear operator connected to the q-binomial theorem, we are able to provide two new subclasses of analytic functions that are applicable to the open symmetric unit disc. In addition to this, we go into inclusion relations and attributes that integral operators must preserve for functions that fall within these classes. This study generalises certain previously discovered findings and also includes some newly discovered ones.

Keywords: *q-difference operator; q-binomial theorem; star like and convex functions; ruscheweyh differential operator; inclusion relations; q-bernardi integral operator.*

Introduction:

At the turn of the previous century, Frank Hilton Jackson was responsible for the methodical development of q-calculus as a direct consequence of the groundbreaking work done by Euler and Heine. Jackson is credited with developing the ideas of the q-derivative (Jackson [1]) and the q-integral during the course of his work (Jackson [2]). To put it another way, q-calculus is just regular old classical calculus but without the limits concept. There are several applications for the symmetric q-calculus, particularly in the realm of quantum physics; for examples, see [3,4]. In addition, the field of q-calculus is undergoing brisk expansion as a direct result of the many applications it has in the fields of mathematics, mechanics, and physics. This history of q-calculus can be illustrated by the wide variety of applications it has had in fields such as quantum mechanics, analytic number theory, theta functions, hypergeometric functions, finite difference theory, gamma function theory, Bernoulli and Euler polynomials, mock theta functions, combinatorics, multiple hypergeometric functions, Sobolev spaces, operator theory, and, more recently, in analytic and harmonic univalent functions. The use of q-calculus to approximation theory was pioneered by Lupas [5], and q-Bernstein polynomials are one of its results. Ismail et al. [6] were the first to apply q-calculus to geometric function theory. They did this by generalising the set of starlike functions into a q-analogue, which they referred to as the set of q-starlike functions (GFT). The work of Srivastava, which can be found referenced in [7] and examined the operators of q-calculus and fractional q-calculus as well as their applications in the generalised Fourier transform of complex analysis, was also important in this regard. In the same vein as the previous proposal, the q-difference operator has been the subject of substantial research in the area of GFT by a number of different writers. We suggest you to [8–15] for some recent research papers that are associated with this operator on the classes of analytic functions. The q-series hypothesis was developed as a response to the discovery that

$$\lim_{q\to 1}\frac{1-q^m}{1-q} = m \text{ for } m \in \mathbb{C},$$

where $\mathbb{C}$ is the set of complex numbers. For $0 < |q| < 1$ the number

$$[m]_q := \frac{1-q^m}{1-q}$$

is called a q-number (or basic number). The q-shifted factorial, see [16], is defined for a ∈ C by

$$(a;q)_n = \begin{cases} 1 & \text{if } n = 0 \\ (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1}), & \text{if } n \in \mathbb{N} = \{1,2,\dots\}. \end{cases}$$

It is easy to see that

$$\lim_{q\to 1}\frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n,$$

where $(\alpha)_n$ is the familiar Pochhammer symbol given by

$$(\alpha)_n = \begin{cases} 1 & (n = 0; \alpha \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}) \\ \alpha(\alpha+1)...(\alpha+n-1) & (n \in \mathbb{N}; \alpha \in \mathbb{C}). \end{cases}$$

The following formula is one of the most important summation formulas for hypergeometric series:

$${}_1F_0(a;-;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n = (1-z)^{-a} \quad (|z| < 1).$$

A q-analogue of this formula is called the q-binomial theorem:

$${}_1\Phi_0(a;-;q,z) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty} \quad (|z| < 1),$$

see Gasper and Rahman [17] (p. 8)). Jackson's q-derivative of a function f defined on a subset of $\mathbb{C}$ is given by (see [1,2])

$$D_q f(z) := \frac{f(z) - f(qz)}{(1-q)z} \qquad (z \neq 0).$$

which have a solution that is analytic in the open symmetric unit disc U = z, where z is less than C and |z| is less than 1. The class S q () of q-starlike functions of order, 0 1, was established by Seoudy and Aouf [18]. This class is comprised of all functions f A that meet the inequality.

$$\Re\left(\frac{zD_q f(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}).$$

For the functions $f_j(z)(j = 1, 2)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{n+1,j} z^{n+1},$$

let $f_1 * f_2$ denote the Hadamard product (or convolution) of f_2 and f_2 defined by

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n+1,1} a_{n+1,2} z^{n+1}.$$

In [19], Ruscheweyh introduced an operator $D^n : A \to A$ defined by convolution:

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) \qquad (n > -1, \; z \in \mathbb{U}),$$

which implies that

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \qquad (n \in N_0 := N \cup \{0\}).$$

Ruscheweyh considered the class:

$$R_n = \{f \in A : \Re\left(\frac{D^{n+1} f(z)}{D^n f(z)}\right) > \frac{1}{2}\},$$

We observe that when $q \to 1^-$, we have $R_q^n f(z) = D^n f(z)$. For more details on the q-analogue Ruschewewh differential operators, see [24–27]. Now, we define the function $\varphi(a, q, z)$ by

$$\varphi(a, q; z) = z\,{}_1\Phi_0(a; -; q, z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^{n+1} \quad (z \in \mathbb{U}).$$

Corresponding to the function $\varphi(a, q, z)$, we define a linear operator $L(a, q)$ on A by the convolution

$$L(a, q) f(z) = \varphi(a, q; z) * f(z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} a_{n+1} z^{n+1} \qquad (a_1 = 1).$$

Remark 1. *For $f(z) \in A$*

$$L(q^{\alpha+1}, q) f(z) = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1};q)_n}{(q;q)_n} a_{n+1} z^{n+1} = R_q^{\alpha} f(z).$$

For the operator $L(a, q)$, it is easy to verify the following identity

$$\frac{1-a}{1-q} L(aq, q) f(z) = \frac{q-a}{q(1-q)} L(a, q) f(z) + \frac{a}{q} z D_q L(a, q) f(z). \tag{2}$$

When $a = q^{\alpha+1}$ in (2), we get the identity given by Aldweby and Darus in [23] for the operator R_q^n.

The following definition is a generalization of the definition of the class K_n given by Singh and Singh [20].

Main Results:

In order to prove our main results, we shall require the following lemma to be used in the sequel.

Lemma 1 (q-Jack lemma [40]). *Let $\omega(z)$ be analytic in $\mathbb{U}$ with $\omega(0) = 0$. Then, if $|\omega(z)|$ attains its maximum value on the circle $|z| = r(r < 1)$ at a point z_0, we can write*

$$z_0 D_q \omega(z_0) = k\omega(z_0),$$

where k is real and $k \geq 1$.

Theorem 1. *Let $0 < q < 1$ and $0 \leq a < q$, then*

$$R(aq, q) \subset R(a, q).$$

Proof. Suppose $f(z) \in R(aq, q)$, then

$$\Re\left\{\frac{L(aq^2, q)f(z)}{L(aq, q)f(z)}\right\} > \frac{1-a}{1-aq}. \tag{3}$$

We have to show that (3) implies the following inequality

$$\Re\left\{\frac{L(aq, q)f(z)}{L(a, q)f(z)}\right\} > \frac{q-a}{q(1-a)} \quad .$$

Define $\omega(z)$ in $\mathbb{U}$ by

$$\frac{L(aq, q)f(z)}{L(a, q)f(z)} = \frac{q-a}{q(1-a)} + \frac{a(1-q)}{q(1-a)} \frac{1-\omega(z)}{1+\omega(z)}. \tag{4}$$

Clearly, $\omega(0) = 0$. Equation (4) may be written as

$$\frac{L(aq, q)f(z)}{L(a, q)f(z)} = \frac{q(1-a) + [(q-a) - a(1-q)]\omega(z)}{q(1-a)[1+\omega(z)]}. \tag{5}$$

With the q-derivative rules and some simple calculations, (5) gives

$$\frac{zD_q L(aq, q)f(z)}{L(aq, q)f(z)} - \frac{zD_q L(a, q)f(z)}{L(a, q)f(z)} = -\frac{2a(1-q)zD_q\omega(z)}{[q(1-a) + [(q-a) - a(1-q)]\omega(z)][1+\omega(z)]}.$$

Using the identity (2) and Equation (4), we can conclude that

$$\begin{aligned} \frac{L(aq^2, q)f(z)}{L(aq, q)f(z)} - \frac{1-a}{1-aq} \quad = \quad & \frac{a(1-q)}{(1-aq)} \frac{1-\omega(z)}{1+\omega(z)} \\ & - \frac{2a^2(1-q)^2}{(1-aq)} \frac{zD_q\omega(z)}{[q(1-a) + [(q-a) - a(1-q)]\omega(z)][1+\omega(z)]}. \end{aligned} \tag{6}$$

Conclusion:

The concept of limits is absent from the traditional classical calculus that makes up quantum calculus. Recently, a significant amount of emphasis has been placed by researchers on the subject of q-calculus. The fact that it may be used in a variety of subfields of mathematics and physics is largely responsible for this unusual attention. Jackson [1,2] was one of the first few academics who described the q-analogue of the derivative and integral operators and offered some of their applications. He also provided some examples of how these operators are used. In the field of geometric function theory, many subclasses of normalised analytic functions in the open symmetric unit disc that are related with the q-derivative have previously been explored from a variety of perspectives. These subclasses are found in the open symmetric unit disc. Through the use of a linear operator connected to the q-binomial theorem, we were able to provide two new classes of analytic functions that may be used in the open symmetric unit disc. In addition to this, we spoke about inclusion relations and properties-preserving integral operators for functions that fall within these classes. In addition to presenting some fresh findings, this work also generalises certain previously established findings. It is possible to apply q-calculus to differential subordinations for specific subclasses of analytic functions as preparation for work to be done in the future.

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