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Existence and Uniqueness of Solution of Fractional Differential Equation in Cone Metric Spaces

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Abstract

In this paper we investigate the existence and uniqueness for fractional difference equations in cone metric spaces. The result is obtained by using the some extensions of Banach's contraction principle in complete cone metric space. The theory of strongly continuous cosine family and fractional calculus is used in proving results.

Key words: cone metric space, fixed point, cosine family, fractional calculus.

AMS Mathematics Subject Classification: 47G20, 34K05, 47H10, 34A12.

Introduction

The purpose of this paper is study the existence and uniqueness for fractional and difference equation with classical condition in cone metric spaces. we consider the following fractional evolution equation of the form:

$$\Delta^{\alpha} x(t) = A(t)x(t), \quad t \in J = [0,b], \quad (1.1)$$

$$x(0) = x_0 \quad (1.2)$$

Where $0 < \alpha < 1, A(t)$ is a bounded linear operator on a Banach space X with domain $D(A(t))$, the unknown $x(\cdot)$ takes values in the Banach space X, and x_0 is a given element of X. The operator D^{α} denotes the Capto fractional derivative of order α .

In Section 2, we discuss the preliminaries. Section 3, we deals with study of the fractional difference equation and in Section 4, we give examples to illustrate the application of our results

Preliminaries and Statement of Results

Let us recall the concepts of the cone metric space and we refer the reader to [9, 10, 13, 14] for the more details. Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,

1. P is closed, nonempty and $P \neq \{0\}$;
2. $a, b \in, a, b > 0, x, y \in P \Rightarrow ax + by \in P$;
3. $x \in P \; and - x \in P \Rightarrow x = 0$.

For a given cone $P \in E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$.We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in int\, P$. Where $int\, P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of P.

In the following way, we always suppose E is a real Banach space, P is cone in E with $int\, P \neq \phi$, and $\leq$ is partial ordering with respect to P.

Definition 1 [9]: Let X a nonempty set. Suppose that the mapping $d: X \times X \to E$ satisfies:

$(d_1) 0 \leq d(x,y) \forall\, x, y \in X \; and \; d(x,y) = 0 \; iff \; x = y$;

$(d_2) \; d(x,y) = d(y,x), for \; all \; x, y \in X$;

$(d_3) \; d(x,y) \leq d(x,z) + d(z,y), for \; all \; x, y \in X$,

Then d *is* called a cone metric on X and (X, d) is called a cone metric space. The concept of cone metric space is more general than that of metric space. The following example is a cone metric space.

Example 1 [9]: Let $E = \mathbb{R}^2, p = \{(x,y) \in E: x, y \geq 0\}, x = \mathbb{R},$ and$d: X \times X \to E$ such that $d(x,y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant, and then (X, d) is a cone metric space.

Definition 2: Let X be an ordered space. A function $\Phi: X \to X$ is said to a comparison function if every $x, y \in X, x \leq y,$ implies that $\Phi(x) \leq \Phi(y)$, $\Phi(x) \leq x$ and $\lim_{n \to \infty} ||\Phi^n(x)|| = 0$, for every $x \in X$.

Example 2: Let $E = \mathbb{R}^2, p = \{(x, y) \in E: x, y \geq 0\}$, it is easy to check that $\Phi: E \to E$, with $\Phi(x, y) = (ax, ay)$, for some $a \in (0,1)$ is a comparison function. Also if Φ_1, Φ_2 are two comparison functions over $\mathbb{R}$. than $\Phi(x, y) = (\Phi_1(x), \Phi_2(y))$ is also a comparison function over E.

Let X be a Banach space with norm $\|.\|$. Let $B = c([0, b], Z)$ be the Banach space of all continuous function from $[0, b]$ into Z endowed with supremum norm

$$||x||_\infty = \sup\{\|x(t)\|: t \in [0, b]\}.$$

Let $P = \{(x, y): x, y \geq 0\} \subset E = \mathbb{R}^2$, and define $d(f, g) = (||f - g||_\infty, \alpha||f - g||_\infty)$ for every $f, g \in B$. Then it is easily seen that (B, d) is a cone metric space.

Lemma 2.1 [13]: Let (X, d) be a complete cone metric space, where P is a normal cone with normal constant K. Let $f: X \to X$ be a function such that there exists a comparison function $\Phi: P \to P$ such that

$$d\big(f(x), f(y)\big) \leq \Phi\big(d(x, y)\big).$$

For every $x, y \in X$. Then f has unique fixed point.

Main Result:

We need some basic definitions and properties of fractional calculus which are used in this section**.**

Definition 3.1 [6] Let $\alpha > 0$. The α^{th} fractional caputo like difference is defined by

$$\Delta^\alpha x(t) = \Delta^{-(m-\alpha)}(\Delta^m x(t)) \qquad t \in J$$

Where $m - 1 < \alpha < m$

Definition 3.2[7] Let $\alpha > 0$ the fractional sum of the function x is

$$\Delta^{-\alpha} x(\mathrm{t}) = \sum_{s=0}^{t-1} k^\alpha(t - s)x(s) \qquad t \in J$$

Here

$$k^\alpha(j) = \frac{\Gamma(j + \alpha)}{\Gamma(\alpha)\Gamma(j + 1)}$$

Definition 3.3. The function $x \in B$ satisfy the summation equations

$$x(t) = x_0 + \sum_{s=0}^{t-1} k^\alpha(t - s)A(s)x(s), \qquad t \in J$$

is called the solution of the equation (1.1) – (1.2)

We list the following hypotheses for our convenience:

(H$_1$) A(t) is a bounded linear operator on X for each $t \in [0, b]$, the function t $\to$ A(t) is continuous in the uniform operator topology and there exists a constant K such that

$$M = \max_{t \in [0,b]} ||A(t)||$$

(H$_2$) There exists a comparison function $\phi_1: \mathbb{R}^2 \to \mathbb{R}^2$ such that
$(||x(t) - y(t)||, \alpha||x(t) - y(t)||) \leq \phi_1(d(x, y))$ for every $t \in [0, b]$ and $x, y \in B$.

(H$_3$) $Mk^{\alpha+1}(b) \leq 1$

Theorem 3.1. Assume that hypotheses $(H_1) - (H_3)$ hold. Then the evolution equations (1.1) - (1.2) has a unique solution x on J

Proof. The operator $F: B \to \mathrm{B}$ is defined by

$$Fx(t) = x_0 + \sum_{s=0}^{t-1} k^\alpha(t - s)A(s)x(s), \quad t \in J \qquad (3.2)$$

By using hypothesis (H$_1$)-(H$_3$), we have

$$\begin{aligned}
&(||Fx(t) - Fy(t)||, \alpha||Fx(t) - Fy(t)||) \\
&= (||\sum_{s=0}^{t-1} k^\alpha(t - s)A(s)x(s) - \sum_{s=0}^{t-1} k^\alpha(t - s)A(s)y(s)\ ||, \\
&\quad \alpha||\sum_{s=0}^{t-1} k^\alpha(t - s)A(s)x(s) - \sum_{s=0}^{t-1} k^\alpha(t - s)A(s)y(s)\ ||) \\
&\leq \left(||\sum_{s=0}^{t-1} k^\alpha(t - s)A(s)\big(x(s) - y(s)\big)||, \alpha||\sum_{s=0}^{t-1} k^\alpha(t - s)A(s)\big(x(s) - y(s)\big)||\right) \\
&\leq \left(Mk^\alpha(b) \sum_{s=0}^{t-1} ||x(s) - y(s)||, \alpha\, Mk^\alpha(b) \sum_{s=0}^{t-1} ||x(s) - y(s)||\right) \\
&\leq Mk^\alpha(b) \sum_{s=0}^{t-1} (\|x(s) - y(s\|, \alpha\|x(s) - y(s)\|) \\
&\leq Mk^{\alpha+1}(b)\, \phi_1\left(||x - y||_B, \alpha||x - y||_B\right) \\
&\leq \phi_1(d(x, y)
\end{aligned} \qquad (3.3)$$

for all $x, y \in B$. This implies that $d(Fx, Fy) \leq \phi_1(d(x, y))$, for all $x, y \in B$. Now an application of Lemma 2.1., the operator has a unique point in B. This means that the equation (1.1) − (1.2) has unique solution.

Applications:

In this section we give example to illustrate the usefulness of our results. Let us consider example of fractional initial value problem

$$\Delta^\alpha x(t) = \frac{t}{10} x(t), \quad t \in [0, 2], \quad 0 < \alpha \leq 1 \qquad (4.1)$$

$$x(0) = x_0 \qquad (4.2)$$

Consider a metric $d(x,y) = (\|x-y\|_B, \alpha\|x-y\|_B)$ on $C([0,2],R)$ for $\alpha \geq 0$. Then clearly $C([0,2],R)$ is complete cone metric space. Here

$$A(t) = \frac{t}{10}, \qquad t \in [0,2]$$

Clearly,

$$M = \frac{1}{5}$$

Moreover,

$$Mbk^{\alpha}(b) = \frac{1}{5} \times 2 \times k^{\alpha}(2) \leq 1$$

then all conditions of Theorem 3.1. are satisfied, the problem $(4.1)-(4.2)$ has a unique solution $x \in C([0,2],R)$ on $[0,2]$.

Conclusion:

In this paper, we studied the existence for Fractional difference equation in cone metric spaces and proved that solution of this result is unique. We proved this result by using the some extensions of Banach contraction principle in complete cone metric space. Moreover we also gave application of above result.

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