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Degree of Approximation of Function in the Hölder Metric by Matrix – Cesaro Summability Method

Yamini Yadav[1], U.K.Shrivastava[2], C. S. Rathore[3]
[1] Research Scholar, Department of Mathematics,
Atal Bihari Vajpayee Vishwavidyalaya, Bilaspur (C.G.) India
[2] Department of Mathematics, Govt. E.R.R. P.G. College Bilaspur
Atal Bihari Vajpayee Vishwavidyalaya, Bilaspur (C.G.) India
[3] Department of Mathematics, Jajwalyadev Govt. Girls College, Janjgir
Shaheed Nand Kumar Patel University, Raigarh (C.G.) India
Corresponding Author – Yamini Yadav
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Abstract

In this paper a new theorem established on the degree of approximation of function in the Hölder Metric by Matrix Cesaro - Summability method of its Fourier series.

Keywords –Degree of approximation, Hölder Metric, Matrix – Cesaro Summability method, Fourier series .

Introduction

The degree of approximation of function belonging to the $Lip\,\alpha$, $Lip(\alpha\,,p), Lip(\xi(t),p\,)$ and $W\big(L^p\xi(t)\big)$ using different summability method has determined by the several investigators of its Fourier series.

Alexits (1928) determined the degree of approximation of function of $H_\alpha(0 < \alpha \le 1)$and $0 \le \beta < \alpha.$

Then $\|\sigma_n(f) - f\|_\beta = O(1)\begin{cases} n^{\beta-\alpha} & (0<\alpha<1) \\ \frac{1}{n(logn)^{\beta-1}} & (\alpha = 1)\end{cases}$

Chandra (1988), (1993) determined some results on degree of approximation of functions in Holder Metric. In 2008 Singh and Mahajan (2008) studied error bound of periodic signals in the Holder metric. In 2014 Vishnu Narayan Mishra and KejalKhatri extended the result of Singh and Mahajan in 2008. In 2019 Santosh Kumar Sinha ,U.K.Shrivastava established a new theorem in Holder metric by using (N,Pn) (E,q) means .

In the present work we established a new theorem on degree of approximation of function in the Hölder Metric by Matrix Cesaro - Summability method of its Fourier series by using our previous work (2015).

2. Definition and notations

Let f be 2π periodic function, integrable over $(-\pi,\pi)$ in the sense of Lebesgue, then its Fourier series is given by

$$f(t)\sim\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt) \qquad (2.1)$$

Let $C_{2\pi}$denote the Banach Space of all 2π - periodic continuous function defined on $[\pi, -\pi]$ under sub-norm.
For $0 \le \alpha \le 1$ and some positive constant k the function space H_α is given by the following

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \le k\,|x - y|^\alpha\} \qquad (2.2)$$

The space H_α is a Banach space with the norm$\|.\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{x,y}[\Delta^\alpha f(x,y)] \qquad (2.3)$$

Where $\|f\|_c = \sup_{-\pi\le x\le\pi}|f(x)|$and $\Delta^\alpha f(x,y) = |f(x) - f(y)|/|x-y|^\alpha \;\; x \ne y.$

We shall use the connection that $\Delta^0 f(x,y) = 0.$

The metric induced by norm in (2.3) on H_α is called the Hölder metric .

The degree of approximation $E_n(f)$of a function f:R→R by trigonometric polynomial t_n of degree n is defined by

$E_n(f) = \|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}$Zygmund (12).

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose n^{th}partial sum is given by $S_n = \sum_{k=0}^{\infty} u_k$.

Cesaro means (C,1) of sequence $\{S_n\}$ is given by $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k$. If $\sigma_n \to S$, as $n \to \infty$ then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesaro means (C,1) to S.

Let $T = (a_{n,k})$ be an infinite lower triangular matrix satisfying the conditions of regularity, i.e. $\sum_{k=0}^{\infty} |a_{n,k}| \le M$, a finite constant .

Matrix – Cesaro means T(C_1) of the sequence $\{S_n\}$ is given by

$$t_n = \sum_{k=0}^{\infty} a_{n,n-k}\, \sigma_{n-k} = \sum_{k=0}^{\infty} a_{n,n-k} \frac{1}{n-k+1} \sum_{r=0}^{n-k} S_r .$$

If $t_n \to S$ as n → ∞ , then the sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Matrix Cesaro means T(C_1) method to S.

Important particular cases of Matrix -Cesaro means are :

(i) (N,Pn)C_1 means ,when $a_{n,n-k} = \frac{P_k}{P_n}$, where $P_n = \sum_{k=0}^{\infty} P_k \ne 0$

(ii) (N,Pn)C_1 means ,when $a_{n,n-k} = \frac{P_{n-k}}{P_n}$

(iii) (N,p,q)C_1 means ,when $a_{n,n-k} = \frac{P_k q_{n-k}}{R_n}$, where $R_n = \sum_{k=0}^{\infty} P_k q_{n-k} \ne 0$

We write

$$\emptyset(t) = f(x+t) + f(x-t) - f(x)$$

$$K(n,t) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{a_{n,n-k}}{(n-k+1)} \frac{sin^2(n-k+1)\frac{t}{2}}{sin^2\frac{t}{2}}$$

3. Known Results

Das G, GhoshTulika and Ray B K (1995) studied Degree of approximation of functions in the Hölder metric by (e,c) means.

Theorem-1.$0 < \alpha \le 1$ and $0 \le \beta < \alpha$. Let $f \in H_\alpha$. Then

$$\|e_n(f) - f\|_\beta = O(1) \begin{cases} \frac{\log n}{n^{\beta-\alpha}} & (0 < \alpha - \beta \le \frac{1}{2}) \\ \frac{1}{n^{1/2}} & (\frac{1}{2} < \alpha - \beta \le 1) \end{cases}$$

Mahapatra and Chandra (1982) studied for the Hölder continuous function f to obtain error bounds in Holder norm.

Theorem-2. Let $0 \le \beta < \alpha \le 1$ and let$f \in H_\alpha$. Then for $n > 1$.

$$\|e_n^q(f) - f\|_\beta = O\{(n) 2^{\frac{-(\propto-\beta)}{2}} (\log n)^{\frac{\beta}{\alpha}}\}$$

Again Prem Chandra (1988) generalize his results on Degree of approximation of functions in the Hölder metric.

Theorem-3. Let $0 \le \beta < \alpha \le 1$ and Let $f \in H_\alpha$. Then

$$\|e_n^q(f) - f\|_\beta = O\{n^{\beta-\propto} \log n\}$$

Theorem-4. Binod Prasad Dhakal (2010) determined the degree of approximation of certain function belonging to the $Lip\ \alpha$ class by MatrixCesarosummability method.

Theorem-5. We generalized above result in our previous work (2008).

Let $f: R \to R$ is 2π periodic function belonging to the $W(L^p\xi(t))$class, then its degree of approximation by Matrix -Cesaro Summability mean of Fourier series is given by

$$\|t_n(x) - f(x)\|_p = O\left\{(n+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{n+1}\right)\right\}$$

Provided $\xi(t)$ satisfies the following conditions : -

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|}{\xi(t)}\right)^p \sin^{\beta p} t dt\right\}^{\frac{1}{p}} = O\left(\frac{1}{n+1}\right)$$

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p \sin^{\beta p} t\right\}^{\frac{1}{p}} = O\{(n+1)^{\delta}\}$$

4. Main Theorem

In this paper we established a new theorem of Matrix- Cesaro product summability method in the Holder metric .

Theorem :- For $0 \leq \beta < \alpha \leq 1$ and $f \in H_\alpha$ then for $n > 1$

$$\|t_n(f) - f\|_\beta = o\left[(n+1)^{\beta-\alpha+\frac{\beta}{\alpha}}\right]$$

5. Lemmas

Lemma –I If $\emptyset_x(t)$defined in (2.5) then for $f \in H_\alpha$ and $0 < \alpha \leq 1$ we have

$$\left|\emptyset_x(t) - \emptyset_y(t)\right| = M(|x-y|^\alpha) \quad (5.1)$$

$$\left|\emptyset_x(t) - \emptyset_y(t)\right| = M(|t|^\alpha) \quad (5.2)$$

Lemma –II For $0 < t < \frac{1}{n+1}$ and fact that $\frac{1}{\sin t} \leq \frac{\pi}{2t}$for $0 < t \leq \frac{\pi}{2}$,

$$k(n,t) = O(n+1) \quad (5.3)$$

Proof :-

$$K(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n}\frac{a_{n,n-k}}{n-k+1}\frac{sin^2(n-k+1)\frac{t}{2}}{sin^2\frac{t}{2}}$$

$$= \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}(n-k+1)$$

$$\left(\because sin\, n\theta \leq n\sin\theta \leq n\theta\ for\ 0 < \theta < \frac{1}{n}\right)$$

$$\leq \frac{n+1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}$$

$$= \frac{n+1}{2\pi}$$

$$= O(n+1)$$

Lemma -III For$\frac{1}{n+1} < t < \pi$

$$k(n,t) = O\left(\frac{1}{(n+1)t^2}\right) \quad (5.4)$$

Proof : -

$$K(n,t) = \frac{1}{2\pi}\sum_{k=0}^{n}\frac{a_{n,n-k}}{n-k+1}\frac{sin^2(n-k+1)\frac{t}{2}}{sin^2\frac{t}{2}}$$

$$\leq \frac{1}{2\pi}\sum_{k=0}^{n}\frac{a_{n,n-k}}{n-k+1}\frac{\pi^2}{t^2}$$, by Jordan's lemma

$$= \frac{\pi}{2t^2}\sum_{k=0}^{n}\frac{a_{n,n-k}}{n-k+1}$$

$$= \frac{\pi}{2t^2}O\left(\frac{1}{n+1}\right)$$

$$= O\left(\frac{1}{(n+1)t^2}\right)$$

6. Proof of the main theorem :

The n[th] partial sum $S_n(x)$ of the Fourier series (2.1) is given by

$$S_n(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \emptyset(t)\frac{sin\left(n+\frac{1}{2}\right)}{sin\frac{t}{2}}dt \tag{6.1}$$

The (C, 1) transform i.e.σ_n of S_n is given by

$$\frac{1}{n+1}\sum_{k=0}^{n}(S_k(x) - f(x)) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\emptyset(t)}{sin\frac{t}{2}}\sum_{k=0}^{n} sin\left(k+\frac{1}{2}\right)t\ dt$$

$$\sigma_n(x) - f(x) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\emptyset(t)\frac{sin^2(n+1)\frac{t}{2}}{sin^2\frac{t}{2}}dt \tag{6.2}$$

The matrix means of the sequence $\{\sigma_n\}$ is given by

$$\sum_{k=0}^{n} a_{n,k}\big(\sigma_k(x) - f(x)\big) = \int_0^{\pi}\emptyset(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(k+1)}\frac{sin^2(k+1)\frac{t}{2}}{sin^2\frac{t}{2}}dt$$

$$\sum_{k=0}^{n} a_{n,k}\big(\sigma_{n-k}(x) - f(x)\big) = \int_0^{\pi}\emptyset(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(n-k+1)}\frac{sin^2(n-k+1)\frac{t}{2}}{sin^2\frac{t}{2}}dt$$

$$t_n(x) - f(x) = \int_0^{\pi}\emptyset(t)K(n,t)d\,t \tag{6.3}$$

$$= \int_0^{\frac{1}{n+1}}\emptyset(t)K(n,t)d\,t + \int_{\frac{1}{n+1}}^{\pi}\emptyset(t)K(n,t)d\,t$$

$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}.\right]\emptyset(t)K(n,t)dt \tag{6.4}$$

Now $E_n(x) = |t_n(f) - f(x)|$

and $E_n(x,y) = |E_n(x) - E_n(y)|$

$$=\left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}.\right]\left|\emptyset_x(t) - \emptyset_y(t)\right||K(n,t)|dt$$

$$= I_1 + I_2 \text{ ,say} \tag{6.5}$$

Again, $I_1 = \int_0^{\frac{1}{n+1}}\left|\emptyset_x(t) - \emptyset_y(t)\right||K(n,t)|d\,t$

Using Lemma (3.1) and (3.2), we have

$$= O(n+1)\int_0^{\frac{1}{n+1}} t^{\alpha}d\,t$$

$$= O(n+1)\left(\frac{1}{n+1}\right)^{\alpha+1}$$

$$I_1 = O(n+1)^{-\alpha} \tag{6.6}$$

Now

$$I_2 = \int_{\frac{1}{n+1}}^{\pi}\left|\emptyset_x(t) - \emptyset_y(t)\right||K(n,t)|dt$$

Using Lemma (3.1) and (3.3), we have

$$= O(n+1)\int_{\frac{1}{n+1}}^{\pi} t^{\alpha}\left(\frac{1}{t^2}\right)dt = O\left(\frac{1}{n+1}\right)\int_{\frac{1}{n+1}}^{\pi} t^{\alpha-2}dt$$

$$= O\left(\frac{1}{n+1}\right)\left[\frac{t^{\alpha-1}}{\alpha-1}\right]_{\frac{1}{n+1}}^{\pi}$$

$$= O\left(\frac{1}{n+1}\right)\left[\frac{1}{n+1}\right]^{\alpha-1}$$

$$= O(n+1)^{-\alpha} \qquad (6.7)$$

Again $I_1 = \int_0^{\frac{1}{n+1}} |\emptyset_x(t) - \emptyset_y(t)||K(n,t)|d\,t$

$$= O[|x-y|^{\alpha}(n+1)] \qquad (6.8)$$

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} |\emptyset_x(t) - \emptyset_y(t)||K(n,t)|dt$$

$$= O|x-y|^{\alpha} \int_{\frac{1}{n+1}}^{\pi} |K(n,t)|dt$$

$$= O|x-y|^{\alpha} \int_{\frac{1}{n+1}}^{\pi} \frac{1}{(n+1)t^2} dt$$

$$= O|x-y|^{\alpha} \frac{1}{(n+1)} \int_{\frac{1}{n+1}}^{\pi} \frac{1}{t^2} dt$$

$$= O|x-y|^{\alpha} \frac{1}{(n+1)} \left[\frac{1}{-t}\right]_{\frac{1}{n+1}}^{\pi}$$

$$= O|x-y|^{\alpha} \frac{1}{(n+1)}.(n+1)$$

$$= O|x-y|^{\alpha} \qquad (6.9)$$

Now $I_r = {I_r}^{1-\frac{\beta}{\alpha}} {I_r}^{\frac{\beta}{\alpha}}$, where $r = 1,2,3$ ...

From (6.6) and (6.8) we get

$$I_1 = O\left[\{(n+1)^{-\alpha}\}^{1-\frac{\beta}{\alpha}}\{|x-y|^{\alpha}(n+1)\}^{\frac{\beta}{\alpha}}\right]$$

$$= O\left[(n+1)^{\beta-\alpha}\left\{|x-y|^{\beta}(n+1)^{\frac{\beta}{\alpha}}\right\}\right]$$

$$= O\left[(n+1)^{\beta-\alpha+\frac{\beta}{\alpha}}\,|x-y|^{\beta}\right] \qquad (6.10)$$

From (6.7) and (6.9) we get

$$I_2 = O\left[\{(n+1)^{-\alpha}\}^{1-\frac{\beta}{\alpha}}\{|x-y|^{\alpha}\}^{\frac{\beta}{\alpha}}\right]$$

$$= O\left[(n+1)^{\beta-\alpha}\,|x-y|^{\beta}\right] \qquad (6.11)$$

Now form (6.10) and (6.11) we get

$$|f(x)-f(y)| = O\left[(n+1)^{\beta-\alpha+\frac{\beta}{\alpha}}\,|x-y|^{\beta}\right] + O\left[(n+1)^{\beta-\alpha}\,|x-y|^{\beta}\right]$$

$$= O\left[(n+1)^{\beta-\alpha+\frac{\beta}{\alpha}}\,|x-y|^{\beta}\right]$$

and $\Delta^{\beta}[f(x,y)] = \frac{|f(x)-f(y)|}{|x-y|^{\beta}}$ $(x \neq y)$

$$= O\left[(n+1)^{\beta-\alpha+\frac{\beta}{\alpha}}\right] \qquad (6.12)$$

Now

$\|f\|_c = O[(n+1)^{-\alpha}]$ (6.13)

By using (6.12) and (6.13) we get

$$\|t_n(f) - f\|_{\beta} = O\left[(n+1)^{\beta-\alpha+\frac{\beta}{\alpha}}\right]$$

7. Corollary and Examples :

Corollary: The case $\beta = 0$ in the theorem , we can find the following result :

Let $f \in H_\alpha$ and $0 < \propto \leq 1$, then we get

$$\|t_n(f) - f\|_\beta = O[(n+1)^{-\alpha}]$$

Example 1: Let for real p,

$f(p) = \sum_{n=1}^{\infty} \frac{1}{(nlogn)^{3/2}} \sin n\,(p + \log n)$.

Then $\in H_\alpha$, by Zygmund [12]

Example 2: Let for real p and every positive number $'r'$.

$$f_r(p) = \sum_{n=1}^{\infty} \frac{1}{n^{(\frac{1}{2}+\propto)}} sin\{n(u + rlogn)\}$$

where $(0 < \propto < 1)$.

Then f_r is 2π -periodic function also non –linear and $f_r \in H_\alpha$.

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