International Journal of Advance and Applied Research
www.ijaar.co.in
ISSN – 2347-7075 Impact Factor – 7.328
Peer Reviewed Bi-Monthly
Vol.10 No.3 January – February 2023

CHARACTERIZATION OF A-DISTRIBUTIVE SEMILATTICES

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DOI - 10.5281/zenodo.7678150

Abstract:

In this paper, we defined a-distributive semilattice and obtain properties of a-distributive semilattice in terms of a-ideals and â-filter. Also we have obtain several characterizations of a-distributive semilattice.

Key Words: *a-distributive semilattice , a-ideals , â-filter*

Introduction:

0-distributive lattice concept introduced by Varlet. Then P. Balasubramani and P. Venkatanarasimhan have obtain many characterizations with the help of ideal, filter , prime ideal , minimal prime ideal etc. A lattice L with 0 is called a 0-distributive lattice if for all a, b, c ∈ L with $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$. Any distributive lattice with 0 is 0 - distributive. In this paper we will study the a-distributive meet Semilattices. Y. S. Pawar and M. V. Patil introduced the concept a - distributive lattice for any fixed element $a \neq 1$ in bounded lattice. Also defined a-ideal, prime a-ideal, minimal prime a-ideal and â-filter, etc. and obtain many characterization. Any a - distributive lattice is 0 – distributive. In this paper we generalized concept a-distributive lattices to a-distributive semi lattices. Also we introduced a-ideal, prima-ideal, minimal prime a-ideal and â-filter in a-distributive semi lattices.

Let S be a meet semilattice with 0. Let a, b, c in S be such that whenever $\mathbf{b \vee c}$ exists, $\boldsymbol{a \wedge b = 0}$ and $\mathbf{a \wedge c = 0}$ imply $\mathbf{a \wedge (b \vee c) = 0}$, then S is called 0-distributive semilattice. We generalized a-distributive semi lattice as x, y, z, a ($a \neq 1$) in S be such that whenever $\mathbf{y \vee z}$ exists, $\mathbf{x \wedge y \leq a}$ and $\mathbf{x \wedge z \leq a}$ imply $\mathbf{x \wedge (y \vee z) \leq a}$.

The Hasse figure given below is example of a-Distributive Semilattice.

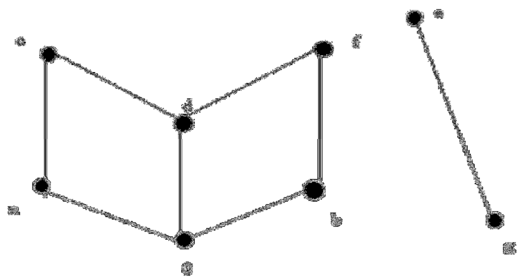

Fig. 1

The following figure shows that a-distributive semilattice need not be distributive.

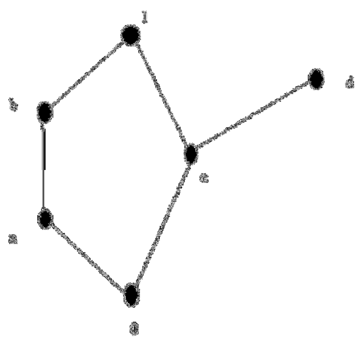

Fig. 2

The following figure is example of semilattice is not a- distributive.

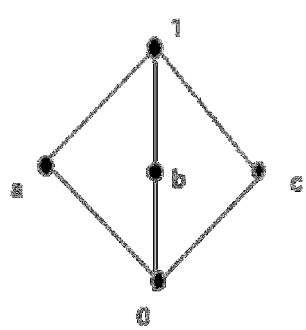

Fig. 3

An ideal I in S is a non-empty subset of S such that a ≤ b, b∈I implies a∈I and whenever avb exists for a, b in I then a v b∈I. (see Venkatanarasimhan [1]). A proper ideal I in S is called prime if x ∧ y ∈I implies that either x ∈ I or y ∈I.

Varlet [5] has generalized the concept of maximal filters and introduce a - maximal filter in L. Thus maximal â - filter in M is a filter in L which is maximal with respect to not containing the given fixed element a (≠ 1) in L . An â - filter is a filter in L not containing a . In this chapter we introduce the concepts of semi a - ideal , a - ideal , prime semi a - ideal and minimal prime a - ideal etc. in S. We begin with simple but essential concepts.

Definition and Properties:

Definition 2. 1: - A non-empty subset I of S is semi- ideal in L if x ≤ y , y ∈ I imply x ∈ I for x , y in S .

Definition 2.2: - A semi-ideal in S containing the element a is called semi a - ideal.

Definition 2. 3: - A prime semi-ideal in S containing the element a is called prime semi a - ideal.

Definition 2.4: - An ideal in S which is maximal w. r. t containing the element a is called maximal a - ideal.

Definition 2.5:- Minimal element in the set of all prime a - ideals in S is called minimal prime a - ideal.

Note that for a = 0 in particular, the above definitions 1 to 5 coincide with the usual definitions of semi-ideal , prime semi-ideal , maximal ideal , minimal prime ideal respectively (see Venkatanarasimhan [1]) .

We begin with a rather elementary result the easy proof of which is omitted.

While proving the properties of prime semi ideals , Venkatanarasimhan [1] has proved that , a nonempty subset F (F ≠ 1) of L is filter if only if (L \F) is prime semi - ideal .

More generally, we prove

Theorem 2.1: - A non - empty subset F of S (F ≠ 1) is â - filter if and only if (S \ F) is prime semi a - ideal .

Proof:- only if part .

let F be â - filter in S . As a ∉ F we get a ∈ (S\ F). Hence (S\ F) is non-empty . Let x ≤ y and y ∈ (S\ F). Suppose x ∉ (S\ F) we get x ∈ F. But as F is filter we get y ∈ F; a contradiction. Therefore x ∈ (S \ F) . Hence (S \ F) is semi a - ideal in S .

If x ∧ y ∈ (S \ F) then x ∧ y ∉ F . As F is filter , either x ∉ F or y ∉ F . Thus x ∈ (S \ F) or y ∈ (S \ F) . This shows (S \ F) is prime semi a - ideal.

If part.

Let (S \ F) be prime semi a - ideal in 1 . To prove that F is â - filter .

(i) As a ∈ (S \ F) we get a ∉ f and hence F is non-empty .

(ii) Let x ≤ y and x ∈ F .Suppose y ∉ F. Then we get y ∈ (S \ F). But as (S \ F) is semi ideal we get x ∈ (S \ F); a contradiction. Thus y ∈ F.

(iii) Let x , y ∈ f . Then x , y ∉ (S \ F) . As (S \ F) is prime semi ideal we get x ∧ y ∉ (S \ F) .i.e. x ∧ y ∈ F. From (i) , (ii) and (iii) we get F is filter in S . As a ∉ F we get F is â - filter.

Theorem 2.2:- Any â - filter in S is contained in some maximal â - filter .

Proof: - Let F be â - filter in S. Define Ꝁ = { J | J is an â - filter in L containing F } . As F ∈ Ꝁ we get Ꝁ is non-empty. Let ξ be any chain in Ꝁ and X = ∪ C ∈ξ C . Then obviously, X is filter in S as X is union of members of chain of filters in S. Further as F ⊆ X and a ∉ X.

We get X ∈ Ꝁ. By Zorn's Lemma, there exists a maximal element M in Ꝁ. This M is maximal â - filter containing F.

Theorem 2.3:- Let F be â - filter in S. Then F is maximal â - filter if and only if for x ∉ F there exists y ∈ F such that x ∧ y ≤ a.

Proof: - Only if Part.

Let F be maximal â – filter in S and x ∉ F.

Then $F \vee [x)$ is filter such that $F \subset F \vee [x)$. But as F is maximal â - filter we get $a \in F \vee [x)$. This implies $a \geq x \wedge y$ for some $y \in F$ and the result follows.

If Part.

Let F be any â - filter in S satisfying the condition in the statement. Now we prove F is maximal â - filter in S. Let if possible there exists â - filter J in L such that $F \subset J \subseteq S$. As $F \subset J$, there exists $x \in J$ such that $x \notin F$. By assumption, there exits $y \in F$ such that $x \wedge y \leq a$. Now $F \subset J$ and $y \in F$ imply $y \in J$. As $x \in J$, $y \in J$ we get $a \in J$; a contradiction. Hence F is maximal â - filter in S.

Characterizations:-

In the following theorem we characterize a - distributive lattices in terms of a - ideals in S.

Theorem 3. 1:- The following statements are equivalent in S.

1. S is a - distributive semi lttice.

2. If $x, y_1, y_2, \ldots, y_n$ in L such that $x \wedge y_i \leq a, \forall i, 1 \leq i \leq n$, then $x \wedge [y_1 \vee y_2 \vee \ldots \vee y_n] \leq a$ if $y_1 \vee y_2 \vee \ldots \vee y_n$ exists.

3. If A is a - ideal and $\{A_i \mid i \in I\}$ is a family of a - ideals such that $A \cap A_i \subseteq (a]$, for all $i, 1 \leq i \leq n$, then $A \cap [\vee_{i \in I} A_i] \subseteq (a]$.

Proof:-

(1) $\Rightarrow$ (2)

As S is a - distributive, the result is true for n = 2. Using the induction on n, the implication follows.

(2) $\Rightarrow$ (3) let a be a - ideal and $\{a_i \mid i \in i\}$ be a family of a - ideals such that

$A \cap A_i \subseteq (a]$, for all $i, 1 \leq i \leq n$. If $x \in a \cap [\vee_{i \in i} a_i]$, then $x \in a$ and $x \in \vee_{i \in i} A_i$. As $x \in \vee_{i \in i} A_i$ we get $x \leq y_1 \vee y_2 \vee \ldots \vee y_n$ for some finite n with $y_i \in a_i$ for all $i, 1 \leq i \leq n$ if $y_1 \vee y_2 \vee \ldots \vee y_n$ exits. As $x \in A$ and $y_i \in A_i$ we get $x \wedge y_i \in A \cap A_i \subseteq (a]$.

Therefore $x \wedge y_i \leq a$, for all $i, 1 \leq i \leq n$.

Hence by assumption (2), $x \wedge [y_1 \vee y_2 \vee \ldots \vee y_n] \leq a$.

Thus $x \leq a$. This shows $A \cap [\vee_{i \in I} A_i] \subseteq (a]$ and the implication follow.

(3) $\Rightarrow$ (1)

To prove that S is a - distributive. Let $x \wedge y \leq a$, $x \wedge z \leq a$ for x, y, z in S. But this turn imply $(x] \wedge (y] \subseteq (a]$ and $(x] \wedge (z] \subseteq (a]$. By assumption (3), we get $(x] \wedge ((y] \vee (z]) \subseteq (a]$. Thus $x \wedge (y \vee z) \leq a$ as $(x] \wedge ((y] \vee (z]) = (x \wedge (y \vee z)]$.

Hence S is a - distributive lattice.

Thus (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) shows that all the statements are equivalent .

Theorem 3. 2:- S is a - distributive if and only if every maximal â - filter is prime.

Proof: - Only if Part .

Let S be a - distributive semilattice and M be any maximal â - filter .

To prove M is prime . Let if possible there exist x , y in L such that $x \vee y \in M$ with $x \notin M$ and $y \notin M$. As M is maximal â - filter, $a \in M \vee [x)$ and $a \in M \vee [y)$. Then $a \geq m_1 \wedge x$ and $a \geq m_2 \wedge y$ for some $m_1, m_2 \in M$. But then $a \geq m_1 \wedge m_2 \wedge x$ and $a \geq m_1 \wedge m_2 \wedge y$ will imply $a \geq (m_1 \wedge m_2) \wedge (x \vee y)$ by a-distributive of S . But as $m_1 \wedge m_2 \in M$ and $x \vee y \in M$ we get $a \in M$; a contradiction. Thus $x \vee y \in M$ must imply $x \in M$ or $y \in M$. This proves that M is prime.

If Part.

Assume that every maximal â - filter in S is prime. To prove that S is

a - distributive semi lattice. Let if possible there exist x , y , z in S such that $x \wedge y \leq a$, $x \wedge z \leq a$ with $x \wedge (y \vee z) \nleq a$. Define $F = [x \wedge (y \vee z))$. Obviously, F is â - filter in S . By Theorem 1.2.7, F is contained in some maximal â - filter say M .But then $x \wedge (y \vee z) \in M$ imply $x \in M$ and $y \vee z \in M$ .M being prime , we get $x \wedge y \in M$ or $x \wedge z \in M$.

But in either the case $a \in M$; a contradiction. Hence $x \wedge y \leq a$, $x \wedge z \leq a$ imply $x \wedge (y \vee z) \leq a$ for x , y , z in S . Hence L is a - distributive lattice.

□□□

A lattice L is distributive if and only if for $x < y$ in L there exists a prime filter F containing x but not y . (see Gratzer [2] , page 78) . In the following theorem we prove a similar characterization for a - distributive semi lattice.

Theorem 3. .3:- L is a - distributive if and only if for $x \nleq a$ in L , there exists prime â - filter containing x .

Proof:- Only if Part .

Let L be a - distributive lattice and $x \nleq a$ for some x in S. As [x) is â - filter . By Theorem 1.2.7, [x) is contained in maximal â - filter say M. S being a - distributive, M is prime â - filter. This shows the existence of prime â - filter M containing x.

If Part .

Suppose L is not a - distributive. Hence there exist x , y , z in L such that $x \wedge y \leq a$, $x \wedge z \leq a$ with $x \wedge (y \vee z) \nleq a$. By hypothesis, there exits prime â - filter P containing $x \wedge (y \vee z)$. Then we get $x \wedge y \in P$ or $x \wedge z \in P$; P being prime â - filter . But in either the case $a \in P$, contradicting the choice of P.

Thus x $\wedge$ y $\leq$ a , x $\wedge$ z $\leq$ a must imply x $\wedge$ (y $\vee$ z) $\leq$ a for all x , y , z in L . Hence S is a – distributive semi lattice.

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