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The Fixed Point Theorems Via Best Approximation

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Abstract:

The theory of fixed point is one of the most important and powerful tools of the modern mathematics not only it is used on a daily bases in pure and applied mathematics but it is also solves a bridge between analysis and topology and provide a very fruitful are of interaction between the two.Fixed point theorems have numerous applications in mathematics. Most of the theorems ensuring the existence of solutions for differential, integral, operator or other equations can be reduced to fixed point theorems.

Introduction :

The fundamental result in the best approximation theory was given by Meinaradus [11], afterwards in (1969), Brosowski [1] theorem has been a basic important result, many authors have studied the applications of fixed point theorem to best approximation theory. Subrahmanyan [21], S.P. Singh [18], M.L.Singh [18], Carbone ([2] [3]), Sahab, Khan and Sessa [14], Hicks and Humphries [15], In (1988), Sahab, Khan and Sessa [14] generalized the result of Singh [19], Recently Pathak, Cho-kang [13] gave an applications of Jungck's [9], fixed point theorem to best approximation theory they extended the result of Singh [19] and Sahab et. al. [14]. In Section 1.2, we have given necessary definitions and results which will be useful in the sequel. In Section 1.3, we have proved some common fixed point theorems of Gregus type in Banach spaces and give application of our fixed point theorem to best approximation theory.

1.2 Preliminaries :

The following definitions will be used in this chapter.

1.2.1 Definition :

Let C be a subset of normed linear space X. Then

(i) A mapping $T : X \to X$ is said to be **contractive** on X. if $\|Tx - Ty\| \le \|x - y\|$ for all *x*, *y* in *X* (resp. *C*)

(ii) The set D_a of best **(*C*, *a*)- approximants** to $\bar{x}$ consists of the point y in C such that

$a\|y - \bar{x}\| = \inf \{\|z - \bar{x}\| : z \in C\}$, where $\bar{x}$ in a point of X, then for $0 < a \le 1$.

Let *D* denote the set of best **C-approximants** to $\bar{x}$. for *a* = 1, our definition reduces to the Set D of best C-approximants to $\bar{x}$.

(iii) A subset *C* of *X* is said to be **starshaped** with respect to a point $q \in C$ if, for all *x* in *C* and all $\lambda \in [0,1]$, $\lambda x + (1-\lambda)q \in C$.

where the point q is called the **star-centre** of C.

(iv) Clearly **convex set** is starshaped [4] with respect to each of its points, but not conversely.

for an example, the set $C = \{0\} \times [0,1] \cup [1,0] \times \{0\}$ is starshaped with respect to $(0,0) \in C$ as the star-centre of C, but it is not convex.

Throughout this chapter F (T) denotes the set of fixed points of T on X.

By relaxing the linearity of the operator T and conexity of Din the original statement of Brosowski [1], Singh [19] proved the following results.

1.2.2. Theorem :

Let C be a T-invariant subset of a normed linear space *X*. Let $T : C \to C$ be a contractive operator on C and let $\bar{x} \in F(T)$. if $D \subseteq X$ is non empty, compact and starshaped, then $D \cap F(T) \ne \phi$.

In the subsequent paper Singh [19], observed that only non expansiveness of T on $D' = D \cup \{\bar{x}\}$ is necessary. Further, Hicks and Humphries [7] have shown the assumption $T : C \to C$ can be weakened to the condition $T : \partial C \to C$ $y \in C$,, i.e., $y \in D$ is not neceassarily in the interior of *C*, where ∂C denotes the boundary of *C*.

Recently, Sahab, Khan and Sessa [14] generalized Theorem 1.2.2 as in the following.

1.2.3 Theorem:

Let X be a Banach space let $T, I : X \to X$ be operators and C be a subset of X such that $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(I)$. Further, suppose that T and I satisfy

$$\|Tx - Ty\| \le \|Ix - Iy\|$$

for all x, y in D', I is linear, continuous on D and $ITx = TIx$ for x in D if D is non empty, compact and starshaped with respect to a point $q \in F(I)$ and $I(D) = D,$ then $D \cap F(T) \cap F(I) \ne \phi$.

Recall that in (1986), Jungck [8] defined the concept of compatibility of two mappings, which includes weakly commuting mappings (Sessa [15]) as proper sub class.

1.2.4 Definition :

Let X be a normed linear space and let $S, T : X \to X$ be two mappings S and T are said to be compatible if, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to x \in X,$ then

$$\|STx_n - TSx_n\| \to 0 \ as \ n \to \infty$$

In (1998), Jungck and Rhoades [10] introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not be true.

1.2.5 Definition :

A pair of S and T is called **weakly compatible** pair if they commute at coincidence points.

1.2.6 Example :

Consider X = [0,2] with the usual metric d. Define mappings $S, T : X \to X$ by

$Sx = 0$ if $x = 0$, $Sx = 0.15$ if $x > 0$

$Tx = 0$ if $x = 0$, $Tx = 0.3$ if $0 < x \le 0.5$, $Tx = x - 0.35$ if $x > 0.5$

Since S and T commute at coincidence point $0 \in X,$ so S and T are weakly compatible maps to see that S and T are not compatible, let us consider a decreasing sequence $\{x_n\}$ where $x_n = 0.5 + \left(\frac{1}{n}\right), n = 1,2,...$

Then $Sx_n \to 0.15, Tx_n \to 0.15 \ but \ STx_n \to 0.15, TSx_n \to 0.3$ as $n \to \infty$. Thus weakly compatible compatible maps need not be compatible.

1.3 Some Results On Common Fixed Points And Best Approximation :

First of all, we prove a common fixed point theorem of Gregus type for compatible mappings in Banach space. Our Theorem is improvement of results of Gregus [6], Jungck [9], Sharma and Deshpande [16].

Throughout this section, we assume that X is Banach space and C is non empty closed convex subset of X.

Now, we prove our main theorem.

1.3.1 Theorem :

Let S and T be compatible mappings of C into itself satisfying the following condition:

$$\begin{aligned}\|Tx - Ty\| \le a\|Sx - Sy\| &+ b\max\{\|Tx - Sx\|, \|Ty - Sy\|\} \\ &+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\} \\ &+ d\max\left\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}(\|Ty - Sx\| + \|Tx - Sy\|)\right\}\end{aligned} \quad ....(1.1)$$

for all x, y in C where $a,b,c,d>0$, $a+b+c+d=1$ and $a+c+d<\sqrt{a}$ if S is linear and continuous in C and $T(C) \subset S(C).$ Then T and S have a unique common fixed point z in C and T is continuous at z.

Proof :

Consider $x = x_0$ be an arbitarary point in C and choose points x_1, x_2 and x_3 in C such that $Sx_1 = Tx$, $Sx_2 = Tx_1$, $Sx_3 = Tx_2$

This can be done since $T(C) \subset S(C).$ for r = 1,2,3,... leads to (1.1)

$$\|Tx_r - Sx_r\| = \|Tx_r - Tx_{r-1}\|$$
$$\leq a\|Sx_r - Sx_{r-1}\| + b\max\{\|Tx_r - Sx_r\|, \|Tx_{r-1} - Sx_{r-1}\|\}$$
$$+ c\max\{\|Sx_r - Sx_{r-1}\|, \|Tx_r - Sx_r\|, \|Tx_{r-1} - Sx_{r-1}\|\}$$
$$+ d\max\{\|Sx_r - Sx_{r-1}\|, \|Tx_r - Sx_r\|, \|Tx_{r-1} - Sx_{r-1}\|, \frac{1}{2}(\|Tx_{r-1} - Sx_r\| + \|Tx_r - Sx_{r-1}\|)\}$$

which shows that,
since $\|Sx_r - Sx_{r-1}\| = \|Tx_{r-1} - Sx_{r-1}\|$, we have, for r = 1,2,3,...

$$\|Tx_r - Sx_r\| \leq \|Tx_{r-1} - Sx_{r-1}\|. \quad ....(1.2)$$

From (1.1) and (1.2) we have

$$\|Tx_2 - Sx_1\| = \|Tx_2 - Tx\|$$
$$\leq a\|Sx_2 - Sx\| + b\max\{\|Tx_2 - Sx_2\|, \|Tx - Sx\|\}$$
$$+ c\max\{\|Sx_2 - Sx\|, \|Tx_2 - Sx_2\|, \|Tx - Sx\|\}$$
$$+ d\max\{\|Sx_2 - Sx\|, \|Tx_2 - Sx_2\|, \|Tx - Sx\|, \frac{1}{2}(\|Tx - Sx_2\| + \|Tx_2 - Sx\|)\}$$
$$\leq a\|Tx_1 - Sx\| + b\max\{\|Tx - Sx\|, \|Tx - Sx\|\}$$
$$+ c\max\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|\}$$
$$+ d\max\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|, \frac{1}{2}(\|Tx - Tx_1\| + \|Tx_2 - Sx\|)\}$$
$$\leq 2a\|Tx - Sx\| + b\|Tx - Sx\| + 2c\|Tx - Sx\| + d\max\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|,$$
$$\frac{1}{2}(\|Tx - Sx_1\| + \|Tx_1 - Sx_1\| + \|Tx_2 - Sx_2\| + \|Sx_2 - Sx\|)\}$$
$$\leq 2a\|Tx - Sx\| + b\|Tx - Sx\| + 2c\|Tx - Sx\| + d\max\{\|Tx_1 - Sx\|, \|Tx - Sx\|, \|Tx - Sx\|,$$
$$\frac{1}{2}(\|Tx - Sx\| + \|Tx - Sx\| + \|Tx_1 - Sx\|)\}$$
$$\leq 2a\|Tx - Sx\| + b\|Tx - Sx\| + 2c\|Tx - Sx\| + 2d\|Tx - Sx\|$$
$$\leq \{(2a + 2c + 2d) + b\}\|Tx - Sx\| \quad ....(1.3)$$

we shall now define a point

$$z = \left(\frac{1}{2}\right)x_2 + \left(\frac{1}{2}\right)x_3.$$

Since C is convex, $z \in C$ and S being linear

$$Sz = \left(\frac{1}{2}\right)Sx_2 + \left(\frac{1}{2}\right)Sx_3$$
$$= \left(\frac{1}{2}\right)Tx_1 + \left(\frac{1}{2}\right)Tx_2 \quad ....(1.4)$$

It follows from (1.2), (1.3) and (1.4) that

$$\|Sz - Sx_1\| = \left\|\left(\frac{1}{2}\right)Tx_1 + \left(\frac{1}{2}\right)Tx_2 - Sx_1\right\|$$

$$\leq \left(\frac{1}{2}\right)\|Tx_1 - Sx_1\| + \left(\frac{1}{2}\right)\|Tx_2 - Sx_1\|$$

$$\leq \left(\frac{1}{2}\right)\|Tx - Sx\| + \left(\frac{1}{2}\right)\{(2a + 2c + 2d) + b\}\|Tx - Sx\|$$

$$\leq \left(\frac{1}{2}\right)\{1 + (2a + 2c + 2d) + b\}\|Tx - Sx\| \qquad ....(1.5)$$

by (1.2) and (1.4), we have

$$\|Sz - Sx_2\| = \left\|\left(\frac{1}{2}\right)Tx_1 + \left(\frac{1}{2}\right)Tx_2 - Sx_2\right\|$$

$$\leq \left(\frac{1}{2}\right)\|Tx_2 - Sx_2\|$$

$$\leq \left(\frac{1}{2}\right)\|Tx - Sx\|. \qquad ....(1.6)$$

By (1.1) and (1.6) we have

$$\|Tz - Sz\| = \left\|Tz - \left(\frac{1}{2}\right)Tx_1 - \left(\frac{1}{2}\right)Tx_2\right\|$$

$$\leq \left(\frac{1}{2}\right)\|Tz - Tx_1\| + \left(\frac{1}{2}\right)\|Tz - Tx_2\|$$

$$\leq \left(\frac{1}{2}\right)a\|Sz - Sx_1\| + \left(\frac{1}{2}\right)b\max\{\|Tz - Sz\|, \|Tx_1 - Sx_1\|\}$$

$$+\left(\frac{1}{2}\right)c\max\{\|Sz - Sx_1\|, \|Tz - Sz\|, \|Tx_1 - Sx_1\|\}$$

$$+\left(\frac{1}{2}\right)d\max\{\|Sz - Sx_1\|, \|Tz - Sz\|, \|Tx_1 - Sx_1\|,$$

$$\frac{1}{2}(\|Tx_1 - Sz\| + \|Tz - Sx_1\|)\} + \left(\frac{1}{2}\right)a\|Sz - Sx_2\| + \left(\frac{1}{2}\right)b\max\{\|Tz - Sz\|, \|Tx_2 - Sx_2\|\}$$

$$+\left(\frac{1}{2}\right)c\max\{\|Sz - Sx_2\|, \|Tz - Sz\|, \|Tx_2 - Sx_2\|\}$$

$$+\left(\frac{1}{2}\right)d\max\{\|Sz - Sx_2\|, \|Tz - Sz\|, \|Tx_2 - Sx_2\|,$$

$$\frac{1}{2}(\|Tx_2 - Sz\| + \|Tz - Sx_2\|)\}$$

$$\leq \left(\frac{1}{4}\right)a[1 + 2a + 2c + 2d + b]\|Tx - Sx\| + \left(\frac{1}{2}\right)b\max\{\|Tz - Sz\|, \|Tx - Sx\|\}$$

$$+\left(\frac{1}{2}\right)c\max\left\{\frac{1}{2}(1 + 2a + 2c + b + 2d)\|Tx - Sx\|, \|Tz - Sz\|, \|Tx - Sx\|\right\}$$

$$+\left(\frac{1}{2}\right)d\max\left[\frac{1}{2}(1+2a+2c+b+2d)\|Tx-Sx\|,\|Tz-Sz\|,\|Tx-Sx\|,\right.$$
$$\left.\frac{1}{2}\{(2+2a+2c+2d+b)\|Tx-Sx\|+\|Tz-Sz\|\}\right]+\left(\frac{1}{4}\right)a\|Tx-Sx\|$$
$$+\left(\frac{1}{2}\right)b\max\{\|Tz-Sz\|,\|Tx-Sx\|\}+\left(\frac{1}{2}\right)c\max\left\{\frac{1}{2}\|Tx-Sx\|,\|Tz-Sz\|,\|Tx-Sx\|\right\} \quad \leq \lambda\|Tx-Sx\|$$
$$+\left(\frac{1}{2}\right)d\max\left\{\frac{1}{2}\|Tx-Sx\|,\|Tz-Sz\|,\|Tx-Sx\|,\ \frac{1}{2}\left(2\|Tx-Sx\|+\|Tz-Sz\|\right)\right\}$$

….(1.7)

where

$$\lambda=\left(\frac{1}{4}\right)a[2+2a+2c+2d+b]+b+\frac{1}{4}c[1+2a+2c+b+2d]$$
$$+\frac{1}{2}c+\frac{1}{4}d[2+2a+2c+b+2d]+d$$
$$<\frac{1}{4}a(3+\sqrt{a})+\frac{1}{4}c(2+\sqrt{a})+b+\frac{1}{2}c+\frac{d}{4}(3+\sqrt{a})+d$$
$$<\frac{a}{4}+\frac{3a}{4}+b+c+d$$
$$=a+b+c+d=1$$

so we have 0< λ <1.

Since x is an arbitary point in C, from (1.7), it follows that there exists a sequence $\{z_n\}$ in C such that

$$\|Tz_0-Sz_0\|\leq\lambda\|Tx_0-Sx_0\|,$$
$$\|Tz_1-Sz_1\|\leq\lambda\|Tz_0-Sz_0\|,$$
$$-------------$$
$$-------------$$
$$\|Tz_n-Sz_n\|\leq\lambda\|Tz_{n-1}-Sz_{n-1}\|,$$

which yield that

$$\|Tz_n-Sz_n\|\leq\lambda^{n+1}\|Tx_0-Sx_0\|,$$

and so we have

$$\lim_{n\to\infty}\|Tz_n-Sz_n\|=0 \qquad ….(1.8)$$

Setting $K_n=\left\{x\in C:\|Tx-Sx\|\leq\frac{1}{n}\right\}$

for n = 1,2, … then (1.8) shows that

$K_n\neq\phi$ for n = 1,2,…

and $K_1\supset K_2\supset K_3\supset\ldots$

obviously, we have $\overline{TK_n}\neq\phi$ and

$\overline{TK_n}\supset\overline{TK_{n+1}}$ for n = 1,2,…

for any x, y in K_n by (1.1), we have

$$\|Tx-Ty\| \le a\|Sx-Sy\| + n^{-1}b + c\max\{\|Sx-Sy\|, n^{-1}\}$$
$$+ d\max\{\|Sx-Sy\|, n^{-1}, \frac{1}{2}(n^{-1}+\|Sx-Sy\|+n^{-1}+\|Sx-Sy\|)\}$$
$$\le a\|Sx-Sy\| + n^{-1}b + c\max\{\|Sx-Sy\|, n^{-1}\}$$
$$+ d\max\{\|Sx-Sy\|, n^{-1}, (n^{-1}+\|Sx-Sy\|)\}$$
$$\le a(2n^{-1}+\|Tx-Ty\|) + n^{-1}b + c(2n^{-1}+\|Tx-Ty\|) + d(3n^{-1}+\|Tx-Ty\|)$$
$$= [a+c]2n^{-1} + [a+c+d]\|Tx-Ty\| + n^{-1}b + 3n^{-1}d$$

Therefore
$$\|Tx-Ty\| \le n^{-1}\{2[a+c]+b+3d\}(1-a-c-d)^{-1}$$

Thus we have
$$\lim_{n\to\infty} diam(\overline{TK_n}) = \lim_{n\to\infty} diam(TK_n) = 0$$

by cantor's theorem, there exists a point u in C such that
$$\bigcap_{n=1}^{\infty}(\overline{TK_n}) = \{u\}.$$

Since $u \in C$ for each $n = 1,2,\ldots$ there exists a point y_n in TK_n such that
$$\|y_n - u\| < n^{-1}$$

Then there exists a point x_n is kK_n such that
$$\|u - Tx_n\| < n^{-1}$$

and so $Tx_n \to u \ \ as \ \ n \to \infty$.

Since $x_n \in k_n$, we have also
$$\|Tx_n - Sx_n\| < n^{-1}$$

and so $Sx_n \to u \ \ as \ \ n \to \infty$.

Since S is continuous $STx_n \to Su \ and \ SSx_n \to Su \ \ as \ \ n \to \infty$.

Moreover $\|TSx_n - STx_n\| \to 0 \ as \ n \to \infty$.

Since S and T are compatible and $Tx_n \to Sx_n \to u \ \ as \ \ n \to \infty$ thus we have $TSx_n \to Su$.

By (1.1) we have
$$\|Tu-Su\| \le \|Tu-TSx_n\| + \|TSx_n - Su\|$$
$$\le a\|Su-SSx_n\| + b\max\{\|Tu-Su\|, \|TSx_n-SSx_n\|\}$$
$$+ c\max\{\|Su-SSx_n\|, \|Tu-Su\|, \|TSx_n-SSx_n\|\}$$
$$+ d\max\{\|Su-SSx_n\|, \|Tu-Su\|, \|TSx_n-SSx_n\|,$$
$$\frac{1}{2}(\|TSx_n-Su\|+\|Tu-SSx_n\|)\} + \|TSx_n-Su\|$$

Letting $n \to \infty$, we obtain

$$\|Tu - Su\| \le a\|Su - Su\| + b\max\{\|Tu - Su\|, \|Su - Su\|\}$$
$$+ c\max\{\|Su - Su\|, \|Tu - Su\|, \|Su - Su\|\}$$
$$+ d\max\{\|Su - Su\|, \|Tu - Su\|, \|Su - Su\|,$$
$$\frac{1}{2}(\|Su - Su\| + \|Tu - Su\|)\} + \|Su - Su\|$$
$$= (b + c + d)\|Tu - Su\|$$
$$= (1 - a)\ \|Tu - Su\|.$$

So we have $Tu = Su$.

Thus $TSu = STu$ and $TTu = TSu = STu$ since S and T are compatible. Furthermore, we have

$$\|TTu - Tu\| \le a\|STu - Su\| + b\max\{\|TTu - STu\|, \|Tu - Su\|\}$$
$$+ c\max\{\|STu - Su\|, \|TTu - STu\|, \|Tu - Su\|\}$$
$$+ d\max\{\|STu - Su\|, \|TTu - STu\|, \|Tu - Su\|$$
$$\frac{1}{2}(\|Tu - STu\| + \|TTu - Su\|)\}$$
$$= (a + c + d)\|TTu - Tu\|$$

This leads to $\|TTu - Tu\| = 0$ since $(a + c + d) < \sqrt{a}$.

Let $z = Tu = Su$.

Then $Tz = z$ and $Sz = STz = TSz = Tz = z$.

Thus z is a unique common fixed point of T and S. The uniqueness of z is a consequence of inequality (1.1). Now, we show that T is continuous at z. Let $\{y_n\}$be a sequence in C such that $y_n \to z$.

Since S is continuous, $Sy_n \to Sz$, By (1.1), we have

$$\|Ty_n - Tz\| \le a\|Sy_n - Sz\| + b\max\{\|Ty_n - Sy_n\|, \|Tz - Sz\|\}$$
$$+ c\max\{\|Sy_n - Sz\|, \|Ty_n - Sy_n\|, \|Tz - Sz\|\}$$
$$+ d\max\{\|Sy_n - Sz\|, \|Ty_n - Sy_n\|, \|Tz - Sz\|,$$
$$\frac{1}{2}(\|Tz - Sy_n\| + \|Ty_n - Sz\|)\}$$
$$\le a\|Sy_n - Sz\| + b\max\{\|Ty_n - Tz\| + \|Tz - Sy_n\|\}$$
$$+ c\max\{\|Sy_n - Sz\|, \|Ty_n - Tz\| + \|Tz - Sy_n\|\}$$
$$+ d\max\{\|Sy_n - Sz\|, \|Ty_n - Tz\| + \|Tz - Sy_n\|, \|Tz - Sz\|,$$
$$\frac{1}{2}(\|Tz - Sy_n\| + \|Ty_n - Sz\|)\}$$

$$\leq a\|Sy_n - Sz\| + b\{\|Ty_n - Tz\| + \|Sz - Sy_n\|\}$$

$$+ c\{\|Ty_n - Tz\| + \|Sz - Sy_n\|\}$$

$$+ d\{\|Ty_n - Tz\| + \|Sz - Sy_n\|,$$

$$\|Ty_n - Tz\| \leq (a+b+c+d)\|Sy_n - Sz\| + (b+c+d)\|Ty_n - Tz\|$$

$$\leq (a+b+c+d)(1-b-c-d)^{-1}\|Sy_n - Sz\|$$

Therefore, we have $Ty_n \to Tz$ and so T is continuous at *z*.

This completes the proof.

As a consequences of our Theorem 1.3.1, we have the following results.

1.3.1.1Corollary:

Let *S* and *T* be compatible mappings of *C* into itself satisfying the following condition:

$$\|Tx - Ty\| \leq a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$

$$+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\}$$

for all *x, y* in *C* where $a,b,c>0$, $a+b+c=1$ and $a+c<\sqrt{a}$ if *S* is linear and continuous in *C* and $T(C) \subset S(C)$. Then *T* and *S* have a unique common fixed point *z* in *C* and *T* is continuous at *z*.

Corallary 1.3.1.1 shows the result of Sushil Sharma and Bhawna Deshpande [16], which obtain by putting $d = 0$.

Now if b=0, c=0 then we get the following corallary

1.3.1.2 Corollary:

Let *S* and *T* be compatible mappings of C into itself satisfying the following condition:

$$\|Tx - Ty\| \leq a\|Sx - Sy\| + (1-a)\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$

for all *x, y* in *C*, 0 <a <1, if *S* is linear and continuous in *C* and $T(C) \subset S(C)$, Then *T* and *S* have a unique common fixed point *z* in *C* and *T* is continuous at *z*.

1.3.1.3 Remark :

Corallary (1.3.1.2) also proves continuity of *T*, so it improves the result of Jungck [9].

if we put $a = b = c = 0$ then we get the following result

1.3.1.4 Corollary:

Let *S* and *T* be compatible mappings of *C* into itself satisfying the following condition:

$$\|Tx - Ty\| \leq d\max\left\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}(\|Ty - Sx\| + \|Tx - Sy\|)\right\}$$

for all *x, y* in *C* where $0 \leq d < 1$, if *S* is linear and continuous in *C* and $T(C) \subset S(C)$. Then *T* and *S* have a unique common fixed point *z* in *C* and *T* is continuous at *z*.

To demonstrate the validity of our Theorem 1.3.1, we have the following example

Example :

Let $X = R$ *and* $C = [0,1]$ with the usual norm. Consider the mappings T and S on C defined as

$$Tx = \frac{1}{4}x \quad and \quad Sx = \frac{1}{2}x \quad for\ all\ x \in C$$

Then $T(C) = \left[0, \frac{1}{4}\right] \subset S(C) = \left[0, \frac{1}{2}\right]$.

It is easy to see that S is linear and continuous.

Further, *T* and *S* are compatible if $\lim_{n\to\infty} x_n = 0,\ where\ \{x_n\}$ is a sequence in *C* such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0$ for some $0 \in C$.

If we take $a = 1/9, b = 13/18, c = 3/18, d = 0$ we see that the condition (1.1) of our Theorem

1.3.1, is satisfied also we have

$a+b+c=1$ and $a+c<\sqrt{a}$.

Thus all the conditions of Theorem 1.3.1 are satisfied and 0 is the unique common fixed point of S and T.

Now, in our next theorem, we give an application of our fixed point theorem to best approximation theory. We improve the results of Pathak, Cho-Kang [13] and Sharma-Deshpande [16].

1.3.2 Theorem :

Let T and S be mapping of X into itself. Let $T:\partial C\to C$ and $\bar{x}\in F(T)\cap F(S)$. Further, suppose that T and S satisfy the condition (1.1), for all x, y in $D_a'=D_a\cup\{x\}\cup E$, where

$E=\{q\in X: Tx_n, Sx_n\to q, \{x_n\}\subset D_a\}$, $a,b,c,d>0$, $a+b+c+d=1$, $a+c+d<\sqrt{a}$, S is linear, continuous on D_a and T,S are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a\cap F(T)\cap F(S)\neq\phi$.

Proof :

Let $y\in D_a$ and hence Sy is in D_a since $S(D_a)=D_a$.

Further, if $y\in\partial C$, then Ty is in C, since $T(\partial C)\subseteq C$. from (1.1), it follows that

$$\begin{aligned}\|Ty-\bar{x}\| &= \|Ty-T\bar{x}\|\\ &\le a\|Sy-S\bar{x}\|+b\max\{\|Ty-Sy\|,\|T\bar{x}-S\bar{x}\|\}\\ &+c\max\{\|Sy-S\bar{x}\|,\|Ty-Sy\|,\|T\bar{x}-S\bar{x}\|\}\\ &+d\max\{\|Sy-S\bar{x}\|,\|Ty-Sy\|,\|T\bar{x}-S\bar{x}\|,\\ &\tfrac{1}{2}(\|T\bar{x}-Sy\|+\|Ty-S\bar{x}\|)\}\end{aligned}$$

$$\begin{aligned}\|Ty-\bar{x}\| &\le a\|Sy-\bar{x}\|+b\max\{\|Ty-Sy\|,\|\bar{x}-\bar{x}\|\}\\ &+c\max\{\|Sy-\bar{x}\|,\|Ty-Sy\|,\|\bar{x}-\bar{x}\|\}\\ &+d\max\{\|Sy-\bar{x}\|,\|Ty-Sy\|,\|\bar{x}-\bar{x}\|,\ \tfrac{1}{2}(\|\bar{x}-Sy\|+\|Ty-\bar{x}\|)\}\end{aligned}$$

$$\begin{aligned}\|Ty-\bar{x}\| &\le a\|Sy-\bar{x}\|+b\{\|Ty-\bar{x}\|+\|\bar{x}-Sy\|\}\\ &+c\max\{\|Sy-\bar{x}\|,\|Ty-\bar{x}\|+\|\bar{x}-Sy\|\}\\ &+d\max\{\|Sy-\bar{x}\|,\|Ty-Sy\|,\ \tfrac{1}{2}(\|\bar{x}-Sy\|+\|Ty-\bar{x}\|)\}\end{aligned}$$

$$\begin{aligned}\|Ty-\bar{x}\| &\le a\|Sy-\bar{x}\|+b\{\|Ty-\bar{x}\|+\|\bar{x}-Sy\|\}+c\{\|Ty-\bar{x}\|+\|\bar{x}-Sy\|\}\\ &+d\max\{\|Sy-\bar{x}\|,\|Ty-\bar{x}\|+\|\bar{x}-Sy\|,\ \tfrac{1}{2}(\|\bar{x}-Sy\|+\|Ty-\bar{x}\|)\}\end{aligned}$$

$$\|Ty-\bar{x}\| \le a\|Sy-\bar{x}\| + b\|Ty-\bar{x}\| + b\|\bar{x}-Sy\| + c\|Ty-\bar{x}\| + c\|\bar{x}-Sy\|$$

$$+ d\{\|Ty-\bar{x}\| + \|\bar{x}-Sy\|\}$$

$$\|Ty-\bar{x}\| \le (a+b+c+d)\|Sy-\bar{x}\| + (b+c+d)\|Ty-\bar{x}\|$$

$$(1-b-c-d)\|Ty-\bar{x}\| \le \|Sy-\bar{x}\|$$

$$a\|Ty-\bar{x}\| \le \|Sy-\bar{x}\|,$$

which implies $a\|Ty-\bar{x}\| \le \|Sy-\bar{x}\|$ and so *Ty* is in D_a. Thus *T* maps D_a into itself.

Proceeding as in theorem (1.3.1), we can show that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u \qquad \text{….(1.9)}$$

Therefore, for a sequence $\{x_n\}$ in D_a the existence of (1.9) is guaranteed whenever $D_a \subset K_n$. Moreover $u \in E$. Since *S* and *T* are compatible and *S* is continuous, we have

$$\lim_{n\to\infty} TSx_n = Su$$

and $\lim_{n\to\infty} S^2 x_n = Su$

By (1.1), we have

$$\|TSx_n - \bar{x}\| = \|TSx_n - T\bar{x}\|$$

$$\le a\|S^2x_n - S\bar{x}\| + b\max\{\|TSx_n - S^2x_n\|, \|T\bar{x} - S\bar{x}\|\}$$

$$+ c\max\{\|S^2x_n - S\bar{x}\|, \|TSx_n - S^2x_n\|, \|T\bar{x} - S\bar{x}\|\}$$

$$+ d\max\{\|S^2x_n - S\bar{x}\|, \|TSx_n - S^2x_n\|, \|T\bar{x} - S\bar{x}\|$$

$$\frac{1}{2}(\|T\bar{x} - S^2x_n\| + \|TSx_n - S\bar{x}\|)\},$$

which implies, as $n \to \infty$

$$\|Su-\bar{x}\| \le a\|Su-\bar{x}\| + b\max\{\|Su-Su\|, \|\bar{x}-\bar{x}\|\}$$

$$+ c\max\{\|Su-\bar{x}\|, \|Su-Su\|, \|\bar{x}-\bar{x}\|\}$$

$$+ d\max\{\|Su-\bar{x}\|, \|Su-Su\|, \|\bar{x}-\bar{x}\|, \frac{1}{2}(\|\bar{x}-Su\| + \|Su-\bar{x}\|)\}$$

$$\le a\|Su-\bar{x}\| + c\|Su-\bar{x}\| + d\|Su-\bar{x}\|$$

$$\|Su-\bar{x}\| \le \sqrt{a}\ \|Su-\bar{x}\|.$$

Hence $Su = \bar{x}$, By (1.1) again, we have

$$\|Tu-\bar{x}\| = \|Tu-T\bar{x}\|$$
$$\le a\|Su-S\bar{x}\| + b\max\{\|Tu-Su\|, \|T\bar{x}-S\bar{x}\|\}$$
$$+c\max\{\|Su-S\bar{x}\|, \|Tu-Su\|, \|T\bar{x}-S\bar{x}\|\}$$
$$+d\max\{\|Su-S\bar{x}\|, \|Tu-Su\|, \|T\bar{x}-S\bar{x}\|, \frac{1}{2}(\|T\bar{x}-Su\| + \|Tu-S\bar{x}\|)\}$$

which gives, by taking $Su = \bar{x}$

$$\|Tu-\bar{x}\| \le a\|\bar{x}-\bar{x}\| + b\max\{\|Tu-\bar{x}\|, \|\bar{x}-\bar{x}\|\}$$
$$+c\max\{\|\bar{x}-\bar{x}\|, \|Tu-\bar{x}\|, \|\bar{x}-\bar{x}\|\}$$
$$+d\max\{\|\bar{x}-\bar{x}\|, \|Tu-\bar{x}\|, \|\bar{x}-\bar{x}\|, \frac{1}{2}(\|\bar{x}-\bar{x}\| + \|Tu-\bar{x}\|)\}$$
$$\le b\|Tu-\bar{x}\| + c\|Tu-\bar{x}\| + d\|Tu-\bar{x}\|$$
$$\|Tu-\bar{x}\| \le (1-a)\|Tu-\bar{x}\|.$$

So $Tu = \bar{x}$.

Next, we consider

$$\|Tu-Tx_n\| \le a\|Su-Sx_n\| + b\max\{\|Tu-Su\|, \|Tx_n-Sx_n\|\}$$
$$+c\max\{\|Su-Sx_n\|, \|Tu-Su\|, \|Tx_n-Sx_n\|\}$$
$$+d\max\{\|Su-Sx_n\|, \|Tu-Su\|, \|Tx_n-Sx_n\|, \frac{1}{2}(\|Tx_n-Su\| + \|Tu-Sx_n\|)\}$$

Letting $n \to \infty$, we get

$$\|\bar{x}-u\| \le a\|\bar{x}-u\| + b\max\{\|\bar{x}-\bar{x}\|, \|u-u\|\} + c\max\{\|\bar{x}-u\|, \|\bar{x}-\bar{x}\|, \|u-u\|\}$$
$$+d\max\{\|\bar{x}-u\|, \|\bar{x}-\bar{x}\|, \|u-u\|, \frac{1}{2}(\|u-\bar{x}\| + \|\bar{x}-u\|)\}$$
$$\|\bar{x}-u\| \le (a+c+d)\ \|\bar{x}-u\|$$
$$\le \sqrt{a}\ \ \|\bar{x}-u\|, \text{ since } a+c+d < \sqrt{a}$$

and so $\bar{x} = u$, i.e., $u=Su=Tu$. By theorem (1.3.1), u must be unique. Hence E = {u}. Then

$$D_a^1 = D_a \cup \{u\}$$

Let $\{k_n\}$ be a monotonically non-decreasing sequence of real numbers such that $0 \le k_n < 1$ and $\overline{\lim}_{n\to\infty} k_n = 1$. Let $\{x_j\}$be a sequence in D_a^1 satisfying (1.2), for each $n \in N$, define a mapping $T_n : D_a^1 \to D_a^1$ by

$$T_n x_j = k_n T x_j + (1-k_n)p$$

for each $n \in N$, it is possible to define such a mapping T_n. Since D_a' is starshaped with respect to $p \in F(S)$.

Since S is linear, we have

$$T_n Sx_j = k_n TSx_j + (1-k_n)p,$$

$$ST_n Sx_j = k_n STx_j + (1-k_n)p.$$

By compatibility of *S* and *T*, we have for each $n \in N$

$$0 \le \lim_{j\to\infty} \|T_n Sx_j - ST_n x_j\|$$

$$\le k_n \lim_{j\to\infty} \|TSx_j - STx_j\| + \lim_{j\to\infty} (1-k_n)\|p-p\|$$

and so

$$\lim_{j\to\infty} \|T_n Sx_j - ST_n x_j\| = 0$$

whenever $\lim_{j\to\infty} Sx_j = \lim_{j\to\infty} T_n x_j = u$ since we have

$$\lim_{j\to\infty} T_n x_j = k_n \lim_{j\to\infty} Tx_j + (1-k_n)u$$

$$= k_n u + (1-k_n)u$$

$$= u.$$

Thus *S* and T_n are compatible on D'_a for each *n* and $T_n(D'_a) \subset D'_a = S(D'_a)$.

On the other hand by (1.1), for all $x, y \in D'_a$ we have, for all $j \ge n$ and n fixed

$$\|T_n x - T_n y\| = k_n \|Tx - Ty\|$$

$$\le k_j \|Tx - Ty\|$$

$$< \|Tx - Ty\|$$

$$\le a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$

$$+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\}$$

$$+ d\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}(\|Ty - Sx\| + \|Tx - Sy\|)\}$$

$$\le a\|Sx - Sy\| + b\max\{\|Tx - T_n x\| + \|T_n x - Sx\|, \|Ty - T_n y\| + \|T_n y - Sy\|\}$$

$$+ c\max\{\|Sx - Sy\|, \|Tx - T_n x\| + \|T_n x - Sx\|, \|Ty - T_n y\| + \|T_n y - Sy\|\}$$

$$+ d\max\{\|Sx - Sy\|, \|Tx - T_n x\| + \|T_n x - Sx\|, \|Ty - T_n y\| + \|T_n y - Sy\|,$$

$$\frac{1}{2}(\|Ty - T_n y\| + \|T_n y - Sx\| + \|Tx - T_n x\| + \|T_n x - Sy\|)\}$$

$$\le a\|Sx - Sy\| + b\max\{(1-k_n)\|Tx - p\| + \|T_n x - Sx\|, (1-k_n)\|Ty - p\| + \|T_n y - Sy\|\}$$

$$+ c\max\{\|Sx - Sy\|(1-k_n)\|Tx - p\| + \|T_n x - Sx\|, (1-k_n)\|Ty - p\| + \|T_n y - Sy\|\}$$

$$+ d\max\{\|Sx - Sy\|, (1-k_n)\|Tx - p\| + \|T_n x - Sx\|, (1-k_n)\|Ty - p\| + \|T_n y - Sy\|$$

$$\frac{1}{2}((1-k_n)\|Ty - p\| + \|T_n y - Sx\| + (1-k_n)\|Tx - p\| + \|T_n x - Sy\|)\}$$

Hence for all $j \ge n$, we have

$$\|T_n x - T_n y\| \le a\|Sx - Sy\| + b\max\{(1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|(1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$+ d\max\{\|Sx - Sy\|, (1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|, \quad \text{....(1.10)}$$
$$\frac{1}{2}\left((1-k_j)\|Ty - p\| + \|T_n y - Sx\| + (1-k_j)\|Tx - p\| + \|T_n x - Sy\|\right)\}$$

Thus, since $\overline{\lim}_{j\to\infty} e_j = 1$, from (1.10), for every $n \in N$, we have

$$\|T_n x - T_n y\| \le \overline{\lim}_{j\to\infty}\|T_n x - T_n y\|$$
$$< \overline{\lim}_{j\to\infty}\Big[a\|Sx - Sy\| + b\max\{(1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|(1-k_j)\|Tx - p\| + \|T_n x(x) - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$+ d\max\{\|Sx - Sy\|, (1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|,$$
$$\frac{1}{2}\left((1-k_j)\|Ty - p\| + \|T_n y - Sx\| + (1-k_j)\|Tx - p\| + \|T_n x - Sy\|\right)\}\Big]$$

which implies

$$\|T_n x - T_n y\| = a\|Sx - Sy\| + b\max\{\|T_n x - Sx\|, \|T_n y - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|, \|T_n x - Sx\|, \|T_n y - Sy\|\}$$
$$+ d\max\{\|Sx - Sy\|, \|T_n x - Sx\|, \|T_n y - Sy\|, \frac{1}{2}\left(\|T_n y - Sx\| + \|T_n x - Sy\|\right)\}$$

for all $x.y \in D'_a$, therefore by theorem (1.3.1) for every $n \in N$, T_n and *S* have a unique common fixed point x_n in D'_a, i.e., every $n \in N$, we have

$$F(T_n) \cap F(S) = \{x_n\}.$$

Now, the compactness of D_a ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ which converges to a point in D_a since

$$x_{n_i} = T_{n_i} x_{n_i} = k_{n_i} T x_{n_i} + (1 - k_{n_i})p \quad \text{....(1.11)}$$

and T is continuous, we have as $i \to \infty$ in (1.3.2.3) $z = Tz$ i.e., $z \in D_a \cap F(T)$.

Further, the continuity of *S* implies that

$$Sz = S(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} Sx_{n_i} = \lim_{i\to\infty} x_{n_i} = z$$

i.e., $z \in F(S)$, therefore, we have $z \in D_a \cap F(T) \cap F(S)$ and so.

$$D_a \cap F(T) \cap F(S) \neq \phi$$

This completes the proof.

As a consequence of our Theorem 1.3.2, we have the following result.

1.3.2.1 Corollary:

Let *T* and *S* be mapping of *X* into itself. Let $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that *T* and *S* satisfy.

$$\|Tx - Ty\| \le a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\}, \frac{1}{2}\left(\|Ty - Sx\| + \|Tx - Sy\|\right)\}$$

for all x, y in $D'_a = D_a \cup \{x\} \cup E$, where

$E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D\}$, *a,b,c>0, a+b+c=1, a+c <* $\sqrt{a}$, *S* is linear, continuous on D_a and *T,S* are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \neq \phi$.

This corollary is obtained by putting d = 0 in Theorem 1.3.2 and is the result of Sushil Sharma and Bhavana Deshpande [16] $d = 0$.

Now, we obtain the following result due to Pathak, Cho and Kang [13] by putting $c = d = 0$ in Theorem **1.3.2.**

1.3.2.2 Corollary:

Let T and S be mappings of X into itself. Let $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that T and S satisfy.

$$\|Tx - Ty\| \le a\|Sx - Sy\| + (1-a)\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$

for all x, y in $D'_a = D_a \cup \{x\} \cup E$, where

$E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D\}$, 0 <*a* <1, if *S* is linear, continuous on D_a and *T,S* are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \neq \phi$.

Now, by putting $a = b = c = 0$ in Theorem 1.3.2 we get the following result.

1.3.2.3 Corollary:

Let *T* and *S* be mapping of *X* into itself. Let $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that *T* and *S* satisfy

$$\|Tx - Ty\| \le d\max\left\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}\left(\|Ty - Sx\| + \|Tx - Sy\|\right)\right\}$$

for all *x, y* in $D'_a = D_a \cup \{x\} \cup E$, where

$E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D_a\}$, 0< d <1, if *S* is linear, continuous on D_a and *T,S* are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \neq \phi$.

Now by weakening the compatibility condition in our Theorem 1.3.1 we prove the following result.

1.3.3 Theorem :

Let *S* and *T* be weakly compatible mappings of *C* into itself satisfying the following condition:

$$\begin{aligned}\|Tx - Ty\| \le{}& a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\} \\ &+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\} \\ &+ d\max\left\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}\left(\|Ty - Sx\| + \|Tx - Sy\|\right)\right\}\end{aligned} \quad ....(1.12)$$

for all *x, y* in *C* where *a,b,c,d>0, a+b+c+d*=1 and *a+c+d<* $\sqrt{a}$ if *S* is linear and continuous in *C* and $T(C) \subset S(C)$. Then *T* and *S* have a unique common fixed point *z** in *C* .

Proof :

Proceeding as in Theorem 1.3.1, we can show that $Tx_n \to u$ *and* $Sx_n \to u$ as $n \to \infty$.

Since $T(C) \subset S(C)$, there exists a point $\upsilon \in C$ such that $u = Sv$, then using the condition (1.12), we have

$$\|Tv-u\| \le \|Tv-Tx_{2n}\| + \|Tx_{2n}-u\|$$

$$\le a\|Sv-Sx_{2n}\| + b\max\{\|Tv-Sv\|, \|Tx_{2n}-Sx_{2n}\|\}$$

$$+ c\max\{\|Sv-Sx_{2n}\|, \|Tv-Sv\|, \|Tx_{2n}-Sx_{2n}\|\}$$

$$+ d\max\{\|Sv-Sx_{2n}\|, \|Tv-Sv\|, \|Tx_{2n}-Sx_{2n}\|,$$

$$\frac{1}{2}(\|Tx_{2n}-Sv\| + \|Tv-Sx_{2n}\|)\} + \|Tx_{2n}-u\|$$

Taking the limit as $n \to \infty$ yields

$$\|Tv-u\| = (b+c+d)\|Tv-u\|$$

$$= (1-a)\|Tv-u\|.$$

So we have *Tv* = *u*. Therefore *Tv* = *Sv* = *u*. Since *T* and *S* are weakly compatible, *TSv* = *STv*, i.e., *Tu* = *Su*.

Let $z^* = Tu = Su$. Again the weak compatibility of *T* and *S* implies. $TSu = STu,\ i.e.,$ Tz*=Sz*. Now we show that *z** is a fixed point of *T*.

$$\|Tz^*-z^*\| = \|Tz^*-Tu\|$$

$$\le a\|Sz^*-Su\| + b\max\{\|Tz^*-Sz^*\|, \|Tu-Su\|\}$$

$$+ c\max\{\|Sz^*-Su\|, \|Tz^*-Sz^*\|, \|Tu-Su\|\}$$

$$+ d\max\{\|Sz^*-Su\|, \|Tz^*-Sz^*\|, \|Tu-Su\|, \frac{1}{2}(\|Tu-Sz^*\| + \|Tz^*-Su\|)\}$$

$$= (a+c+d)\|Tz^*-z^*\|$$

Thus $Tz^* = z^*,$ since $(a+c+d) < a^{1/2}$. Hence $Tz^* = z^* = Sz^*$

Finally, in order to prove the uniqueness of *z** suppose that *z** and $z_1{}^*$ are two common fixed points of *T* and *S*, where *z** is not equal to $z_1{}^*$ then by (1.12), we obtain

$$\|z^*-z_1{}^*\| = \|Tz^*-Tz_1{}^*\|$$

$$\le \|Sz^*-Sz_1{}^*\| + b\max\{\|Tz^*-Sz^*\|, \|Tz_1{}^*-Sz_1{}^*\|\}$$

$$+ c\max\{\|Sz^*-Sz_1{}^*\|, \|Tz^*-Sz^*\|, \|Tz_1{}^*-Sz_1{}^*\|\}$$

$$+ d\max\{\|Sz^*-Sz_1{}^*\|, \|Tz^*-Sz^*\|, \|Tz_1{}^*-Sz_1{}^*\|,$$

$$\frac{1}{2}(\|Tz_1{}^*-Sz^*\| + \|Tz^*-Sz_1{}^*\|)\}$$

$$= (a+c+d)\|z^*-z_1{}^*\|$$

Thus z*= $z_1{}^*$ since $a+c+d<\sqrt{a}$.

This complete the proof.

As a consequence of a Theorem 1.3.3, we get the following result after putting *d* = 0

1.3.3.1Corollary:

Let S and T be weakly compatible mappings of C into itself satisfying the following condition.

$$\|Tx-Ty\| \le a\|Sx-Sy\| + b\max\{\|Tx-Sx\|, \|Ty-Sy\|\}$$
$$+c\max\{\|Sx-Sy\|, \|Tx-Sx\|, \|Ty-Sy\|\}$$

for all x, y in C where $a+b+c=1$, $a+c<\sqrt{a}$, if S is linear in C and $T(C)\subset S(C)$, then T and S have a unique common fixed point z^* in C.

Corollary (1.3.3.1) proves the result of Sushil Sharma and Bhavna Deshpande [17] for weaker condition.

1.3.3.2 Corollary:

Let S and T be weakly compatible mappings of C into itself satisfying the following conditions:

$$\|Tx-Ty\| \le a\|Sx-Sy\| + (1-a)\max\{\|Tx-Sx\|, \|Ty-Sy\|\}$$

$$\textit{for all } x, y \textit{ in } C, 0<a<1$$

if S is linear in C and $T(C)\subset S(C)$, then T and S have a unique common fixed point z^* in C.

As a consequence of a Theorem 1.3.3, we get theCorollary(1.3.3.2) by putting $c = d = 0$.

Now by taking $a = b = c = 0$ in Theorem 5.3.3 we get the following result for weaker condition.

1.3.3.3 Corollary:

Let S and T be weakly compatible mappings of C into itself satisfying the following conditions:

$$\|Tx-Ty\| \le d\max\left\{\|Sx-Sy\|, \|Tx-Sx\|, \|Ty-Sy\|, \frac{1}{2}\left(\|Ty-Sx\|+\|Tx-Sy\|\right)\right\}$$

For all x, y in $C, 0<d<1$

if S is linear in C and $T(C)\subset S(C)$, then T and S have a unique common fixed point z^* in C.

Now we give an application of our fixed point Theorem 1.3.3 to best approximation theory. We improve the result of Pathak, Cho-Kang [13] and Sharma-Deshpande [17] for weaker condition.

1.3.4 Theorem :

Let T and S be mapping of X into itself. Let $T:\partial C \to C$ and $\bar{x}\in F(T)\cap F(S)$. Further, suppose that T and S satisfy the condition (5.12), for all x, y in $D_a' = D_a \cup \{x\} \cup E$, where $E=\{q\in X : Tx_n, Sx_n \to q, \{x_n\}\subset D_a\}$, $a,b,c,d>0$, $a+b+c+d=1$, $a+c+d<\sqrt{a}$, S is linear, continuous on D_a and T,S are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T)\cap F(S) \neq \phi$.

Proof :

Let $y\in D_a$ and hence Sy is in D_a since $S(D_a)=D_a$.

Further, if $y\in\partial C$, then Ty is in C, since $T(\partial C)\subseteq C$. from (5.12), it follows that

$$\|Ty-\bar{x}\| = \|Ty-T\bar{x}\|$$
$$\le a\|Sy-S\bar{x}\| + b\max\{\|Ty-Sy\|, \|T\bar{x}-S\bar{x}\|\}$$
$$+c\max\{\|Sy-S\bar{x}\|, \|Ty-Sy\|, \|T\bar{x}-S\bar{x}\|\}$$
$$+d\max\left\{\|Sy-S\bar{x}\|, \|Ty-Sy\|, \|T\bar{x}-S\bar{x}\|, \frac{1}{2}\left(\|T\bar{x}-Sy\|+\|Ty-S\bar{x}\|\right)\right\}$$

$$\|Ty-\bar{x}\| \le a\|Sy-\bar{x}\| + b\max\{\|Ty-Sy\|, \|\bar{x}-\bar{x}\|\}$$

$$+ c\max\{\|Sy-\bar{x}\|, \|Ty-Sy\|, \|\bar{x}-\bar{x}\|\}$$

$$+ d\max\{\|Sy-\bar{x}\|, \|Ty-Sy\|, \|\bar{x}-\bar{x}\|, \frac{1}{2}(\|\bar{x}-Sy\| + \|Ty-\bar{x}\|)\}$$

$$\|Ty-\bar{x}\| \le a\|Sy-\bar{x}\| + b\{\|Ty-\bar{x}\| + \|\bar{x}-Sy\|\}$$

$$+ c\max\{\|Sy-\bar{x}\|, \|Ty-\bar{x}\| + \|\bar{x}-Sy\|\}$$

$$+ d\max\{\|Sy-\bar{x}\|, \|Ty-Sy\|, \ \frac{1}{2}(\|\bar{x}-Sy\| + \|Ty-\bar{x}\|)\}$$

$$\|Ty-\bar{x}\| \le a\|Sy-\bar{x}\| + b\{\|Ty-\bar{x}\| + \|\bar{x}-Sy\|\} + c\{\|Ty-\bar{x}\| + \|\bar{x}-Sy\|\}$$

$$+ d\max\{\|Sy-\bar{x}\|, \|Ty-\bar{x}\| + \|\bar{x}-Sy\|, \ \frac{1}{2}(\|\bar{x}-Sy\| + \|Ty-\bar{x}\|)\}$$

$$\|Ty-\bar{x}\| \le a\|Sy-\bar{x}\| + b\|Ty-\bar{x}\| + b\|\bar{x}-Sy\| + c\|Ty-\bar{x}\| + c\|\bar{x}-Sy\|$$

$$+ d\{\|Ty-\bar{x}\| + \|\bar{x}-Sy\|\}$$

$$\|Ty-\bar{x}\| \le (a+b+c+d)\|Sy-\bar{x}\| + (b+c+d)\|Ty-\bar{x}\|$$

$$(1-b-c-d)\|Ty-\bar{x}\| \le \|Sy-\bar{x}\|,$$

$$a\|Ty-\bar{x}\| \le \|Sy-\bar{x}\|$$

which implies $a\|Ty-\bar{x}\| \le \|Sy-\bar{x}\|$ and so Ty is in D_a. Thus T maps D_a into itself.

Proceeding as in Theorem 1.3.3, we can show that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u \qquad \text{….(1.13)}$$

Therefore, for a sequence $\{x_n\}$ in D_a the existence of (1.13) is guaranteed whenever $D_a \subset k_n$.

Since $T(D_a) \subseteq S(D_a)$ there exists a point $v \in D_a$ such that $u = Sv$. Then using (1.12).

$$\|Tv-u\| \le \|Tv-Tx_{2n-1}\| + \|Tx_{2n-1}-u\|$$

$$\le a\|Sv-Sx_{2n-1}\| + b\max\{\|Tv-Sv\|, \|Tx_{2n-1}-Sx_{2n-1}\|\}$$

$$+ c\max\{\|Sv-Sx_{2n-1}\|, \|Tv-Sv\|, \|Tx_{2n-1}-Sx_{2n-1}\|\}$$

$$+ d\max\{\|Sv-Sx_{2n-1}\|, \|Tv-Sv\|, \|Tx_{2n-1}-Sx_{2n-1}\|$$

$$\frac{1}{2}(\|Tx_{2n-1}-Sv\| + \|Tv-Sx_{2n-1}\|)\} + \|Tx_{2n-1}-u\|$$

Taking the limit as $n \to \infty$ yields

$$\|Tv-u\| \le (b+c+d)\|Tv-u\|$$

$$= (1-a)\|Tv-u\|$$

So we have $Tv = u$. Therefore $Tv = Sv = u$. Since T and S are weakly compatible, then $TSv = STv$, i.e., $Tu = Su$, by (1.13) we have

$$\|Tu-\bar{x}\| = \|Tu-T\bar{x}\|$$

$$\leq a\|Su-S\bar{x}\| + b\max\{\|Tu-Su\|, \|T\bar{x}-S\bar{x}\|\}$$

$$+ c\max\{\|Su-S\bar{x}\|, \|Tu-Su\|, \|T\bar{x}-S\bar{x}\|\}$$

$$+ d\max\{\|Su-S\bar{x}\|, \|Tu-Su\|, \|T\bar{x}-S\bar{x}\|, \frac{1}{2}(\|T\bar{x}-Su\| + \|Tu-S\bar{x}\|)\}$$

$$\leq a\|Tu-\bar{x}\| + b\max\{\|Tu-Tu\|, \|\bar{x}-\bar{x}\|\} + c\max\{\|Tu-\bar{x}\|, \|Tu-Tu\|, \|\bar{x}-\bar{x}\|\}$$

$$+ d\max\{\|Tu-\bar{x}\|, \|Tu-Tu\|, \|\bar{x}-\bar{x}\|, \frac{1}{2}(\|\bar{x}-Tu\| + \|Tu-\bar{x}\|)\}$$

$$\leq a\|Tu-\bar{x}\| + c\|Tu-\bar{x}\| + d\|Tu-\bar{x}\|$$

$$\|Tu-\bar{x}\| \leq (a+c+d)\|Tu-\bar{x}\|$$

So $Tu = \bar{x}$ since $(a+c+d) < \sqrt{a}$, hence $Tu = Su = \bar{x}$

Next, we consider

$$\|Tu-Tx_n\| = \|\bar{x}-u\|$$

$$\|Tu-Tx_n\| \leq a\|Su-Sx_n\| + b\max\{\|Tu-Su\|, \|Tx_n-Sx_n\|\}$$

$$+ c\max\{\|Su-Sx_n\|, \|Tu-Su\|, \|Tx_n-Sx_n\|\}$$

$$+ d\max\{\|Su-Sx_n\|, \|Tu-Su\|, \|Tx_n-Sx_n\|, \frac{1}{2}(\|Tx_n-Su\| + \|Tu-Sx_n\|)\}$$

Letting $n \to \infty$, we get

$$\|\bar{x}-u\| \leq a\|\bar{x}-u\| + b\max\{\|\bar{x}-\bar{x}\|, \|u-u\|\} + c\max\{\|\bar{x}-u\|, \|\bar{x}-\bar{x}\|, \|u-u\|\}$$

$$+ d\max\{\|\bar{x}-u\|, \|\bar{x}-\bar{x}\|, \|u-u\|, \frac{1}{2}(\|u-\bar{x}\| + \|\bar{x}-u\|)\}$$

$$\|\bar{x}-u\| \leq (a+c+d)\ \|\bar{x}-u\|$$

$$\leq \sqrt{a}\ \ \|\bar{x}-u\|. \text{ since } (a+c+d) < \sqrt{a}$$

$\bar{x} = u$, i.e., $u=Su=Tu$. By Theorem (1.3.3), u must be unique. Hence $E = \{u\}$. Then $D'_a = D_a \cup \{u\}$

Let $\{k_n\}$ be a monotonically non-decreasing sequence of real numbers such that $0 \leq k_n < 1$ and $\overline{\lim}_{n\to\infty} k_n = 1$. Let $\{x_j\}$be a sequence in D'_a satisfying (1.13), for each $n \in N$, define a mapping $T_n : D'_a \to D'_a$ by

$$T_n x_j = k_n T x_j + (1-k_n)p.$$

For each $n \in N$ it is possible to define such a mapping T_n. Since D'_a is starshaped with respect to $p \in F(S)$.

we have,

$$\lim_{j\to\infty} T_n x_j = k_n \lim_{j\to\infty} Tx_j + (1-k_n)u$$
$$= k_n u + (1-k_n)u$$
$$= u.$$

Now, $T_n u = Su = u$ and $T_n Su = u = ST_n u$. Therefore S and T_n commute at their coincidence point Thus S and T_n are weakly compatible on D'_a for each n and $T_n(D'_a) \subset D'_a = S(D'_a)$.

On the other hand by (5.12), for all $x, y \in D'_a$ we have, for all $j \geq n$ and n fixed

$$\|T_n x - T_n y\| = k_n \|Tx - Ty\|$$
$$\leq k_j \|Tx - Ty\|$$
$$< \|Tx - Ty\|$$
$$\leq a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\}$$
$$+ d\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|, \frac{1}{2}(\|Ty - Sx\| + \|Tx - Sy\|)\}$$
$$\leq a\|Sx - Sy\| + b\max\{\|Tx - T_n x\| + \|T_n x - Sx\|, \|Ty - T_n y\| + \|T_n y - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|, \|Tx - T_n x\| + \|T_n x - Sx\|, \|Ty - T_n y\| + \|T_n y - Sy\|\}$$
$$+ d\max\{\|Sx - Sy\|, \|Tx - T_n x\| + \|T_n x - Sx\|, \|Ty - T_n y\| + \|T_n y - Sy\|,$$
$$\frac{1}{2}(\|Ty - T_n y\| + \|T_n y - Sx\| + \|Tx - T_n x\| + \|T_n x - Sy\|)\}$$
$$\leq a\|Sx - Sy\| + b\max\{(1-k_n)\|Tx - p\| + \|T_n x - Sx\|, (1-k_n)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|(1-k_n)\|Tx - p\| + \|T_n x - Sx\|, (1-k_n)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$+ d\max\{\|Sx - Sy\|, (1-k_n)\|Tx - p\| + \|T_n x - Sx\|, (1-k_n)\|Ty - p\| + \|T_n y - Sy\|$$
$$\frac{1}{2}((1-k_n)\|Ty - p\| + \|T_n y - Sx\| + (1-k_n)\|Tx - p\| + \|T_n x - Sy\|)\}$$

Hence for all $j \geq n$, we have

$$\|T_n x - T_n y\| \leq a\|Sx - Sy\| + b\max\{(1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$\leq a\|Sx - Sy\| + b\max\{(1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|(1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|\} \quad ....(1.14)$$
$$+ d\max\{\|Sx - Sy\|, (1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|,$$
$$\frac{1}{2}((1-k_j)\|Ty - p\| + \|T_n y - Sx\| + (1-k_j)\|Tx - p\| + \|T_n x - Sy\|)\}$$

Thus, since $\overline{\lim}_{j\to\infty} e_j = 1$, from (1.14), for every $n \in N$, we have

$$\|T_n x - T_n y\| \le \overline{\lim}_{j\to\infty}\|T_n x - T_n y\|$$
$$\le \overline{\lim}_{j\to\infty}\Big[a\|Sx - Sy\|$$
$$+ b\max\{(1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|(1-k_j)\|Tx - p\| + \|T_n x(x) - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|\}$$
$$+ d\max\{\|Sx - Sy\|, (1-k_j)\|Tx - p\| + \|T_n x - Sx\|, (1-k_j)\|Ty - p\| + \|T_n y - Sy\|,$$
$$\frac{1}{2}\left((1-k_j)\|Ty - p\| + \|T_n y - Sx\| + (1-k_j)\|Tx - p\| + \|T_n x - Sy\|\right)\}\Big]$$

which implies

$$\|T_n x - T_n y\| = a\|Sx - Sy\| + b\max\{\|T_n x - Sx\|, \|T_n y - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|, \|T_n x - Sx\|, \|T_n y - Sy\|\}$$
$$+ d\max\{\|Sx - Sy\|, \|T_n x - Sx\|, \|T_n y - Sy\|, \frac{1}{2}\left(\|T_n y - Sx\| + \|T_n x - Sy\|\right)\}$$

for all $x.y \in D'_a$, therefore by Theorem 1.3.3 for every $n \in N$, T_n and S have a unique common fixed point x_n in D'_a, i.e., every $n \in N$, we have

$$F(T_n) \cap F(S) = \{x_n\}$$

Now, the compactness of D_a ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ which converges to a point in D_a since

$$x_{n_i} = T_{n_i} x_{n_i} = k_{n_i} T x_{n_i} + (1 - k_{n_i})p \qquad \text{....(1.15)}$$

and T is continuous, we have as $i \to \infty$ in (1.15) $z = Tz$ i.e., $z \in D_a \cap F(T)$.
Further, the continuity of S implies that

$$Sz = S(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} Sx_{n_i} = \lim_{i\to\infty} x_{n_i} = z$$

i.e., $z \in F(S)$, therefore, we have $z \in D_a \cap F(T) \cap F(S)$ and so.

$$D_a \cap F(T) \cap F(S) \ne \phi$$

This completes the proof.

As a consequence of our Theorem 1.3.4, we have the following result.

1.3.4.1 Corollary:

Let T and S be mapping of X into itself. Let $T : \partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that T and S satisfy.

$$\|Tx - Ty\| \le a\|Sx - Sy\| + b\max\{\|Tx - Sx\|, \|Ty - Sy\|\}$$
$$+ c\max\{\|Sx - Sy\|, \|Tx - Sx\|, \|Ty - Sy\|\}, \frac{1}{2}\left(\|Ty - Sx\| + \|Tx - Sy\|\right)\}$$

for all x, y in $D'_a = D_a \cup \{x\} \cup E$, where

$E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D\}$, $a,b,c>0$, $a+b+c=1$, $a+c < \sqrt{a}$, S is linear, continuous on D_a and T,S are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \ne \phi$.

This corollary is obtained by putting $d = 0$ in Theorem 1.3.4 and is the result of Sushil Sharma and Bhavana Deshpande [17] $d = 0$.

Now, we obtain the following result due to Pathak, Cho and Kang [13] by putting $c = d = 0$ in Theorem 1.3.4.

1.3.4.2 Corollary:

Let T and S be mappings of X into itself. Let $T:\partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that T and S satisfy.

$$\|Tx-Ty\| \le a\|Sx-Sy\| + (1-a)\max\{\|Tx-Sx\|, \|Ty-Sy\|\}$$

for all x, y in $D'_a = D_a \cup \{x\} \cup E$, where $E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D\}$, 0 $<a<1$, if S is linear, continuous on D_a and T,S are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \ne \phi$.

Now, by putting $a = b = c = 0$ in Theorem 1.3.4 we get the following result.

1.3.4.3 Corollary:

Let T and S be mapping of X into itself. Let $T:\partial C \to C$ and $\bar{x} \in F(T) \cap F(S)$. Further, suppose that T and S satisfy

$$\|Tx-Ty\| \le d\max\left\{\|Sx-Sy\|, \|Tx-Sx\|, \|Ty-Sy\|, \frac{1}{2}\left(\|Ty-Sx\| + \|Tx-Sy\|\right)\right\}$$

for all x, y in $D'_a = D_a \cup \{x\} \cup E$, where $E = \{q \in X : Tx_n, Sx_n \to q, \{x_n\} \subset D_a\}$, $0< d <1$, if S is linear, continuous on D_a and T,S are compatible in D_a, if D_a is nonempty, compact, convex and $S(D_a)=D_a$, then $D_a \cap F(T) \cap F(S) \ne \phi$.

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