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From Visualization to Deduction: Understanding of Pythagorean Theorem Among Middle-Stage Students with Van Hiele's Theory

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Abstract:

Despite geometry being consistently essential in the mathematics curriculum, enhancing logical reasoning, deductive skills, and problem-solving abilities, students are still failing to perform well in geometry. The National Education Policy (NEP, 2020) has stressed the importance of conceptual understanding rather than rote memorisation. In this study, a qualitative approach was adopted to investigate eighth-grade students' understanding level of the Pythagorean theorem through the lens of Van Hiele's Theory, from the Darbhanga district of Bihar. A sample of students was chosen based on academic performance, and they performed a self-designed test on the Pythagorean theorem. Technological tools such as GeoGebra were used for visual assistance, followed by individual interviews to assess understanding levels based on Van Hiele's theory. Data analysis revealed that high-achieving students reached all four Van Hiele levels, while medium-achieving students achieved the informal deduction level, and below-average students were limited to visualization. The study advocates for an enhanced emphasis on cultivating conceptual understanding in geometry, fostering profound learning instead of superficial memorisation.

Keywords: Pythagorean Theorem, Right Angle Triangle, GeoGebra, Van Hiele's Theory.

Introduction:

Geometry is a core element of the global mathematics branch, emphasizing the study of shapes, spatial relationships, and object properties (Luneta, 2014; Bora & Ahmed, 2018). The National Curriculum Framework (NCF, 2005) underscores the importance of geometry at various educational levels. A solid grasp of geometric concepts is essential for students to use geometry in everyday situations effectively. Furthermore, the National Education Policy (NEP, 2020) advocates deeply understanding concepts rather than relying on rote memorization.

Studying geometry enhances students' understanding of other areas of mathematical concepts and encourages connections across different areas of mathematics (Mammana & Villiani, 1998; Muschla & Muschla, 2000; NCTM, 2000). It improves logical reasoning, deductive and analytical thinking, and problem-solving skills, which are crucial tools for learning other fundamental skills. Geometry's relevance extends beyond mathematics into fields like trigonometry, measurement, calculus, and algebra. Many professionals, including physicists, engineers, and land surveyors, use geometric principles in their work (Russell, 2014). Furthermore, geometric thinking significantly contributes to the cognitive development of learners across disciplines (Erdogan, Akkaya, & Celebi Akkaya, 2009). Therefore, developing strong spatial and geometric reasoning skills during the foundational and middle stages is essential for a smooth transition to more advanced mathematical studies.

Given the significance of geometry, many studies by Abu & Abidin, (2013), Luneta, (2014), Alex & Mammen, (2016), Armah et al., (2018), and Armah & Kissi, (2019) have examined teachers' content knowledge and pedagogical strategies in teaching this subject, as well as their impact on students' understanding, using Van Hiele's geometric thought theory. Research has also addressed misconceptions in geometric concepts, such as understanding polygons and quadrilaterals among seventh graders (Ozkan & Bal, 2016). Other studies reveal the gap between pre-service teachers' informal and formal understanding of geometric shapes (Ozdemir Erdogan & Dur, 2014) and studies by Monaghan, (2000), Fujita & Jones, (2007), Okazaki & Fujita, (2007), Fujita, (2012), Halat & Yesil Dagli, (2016) highlights the challenges students face when the orientation of shapes like quadrilaterals changes.

Research by Kilic et al. (2007) indicates that fifth-grade students typically reach Van Hiele's visualization and analysis levels in tessellation understanding. However, Ngirishi & Bansilal (2019) found that students in higher grades, such as tenth grade, often encounter difficulties in recognising properties of shapes and grasping their relationships. Similarly, at the upper secondary level, misconceptions about triangles and quadrilaterals are common among students (Atebe & Schafer, 2008). Although Baiduri, Ismail, and Sulfiyah (2020) argue that some capable junior high students can perform well at the level 2, i.e., analysis,, while using Van Hiele's framework teaching quadrilaterals and triangles has proven to foster a deeper understanding.

Furthermore, Sáenz-Ludlow & Athanasopoulou (2008) found that technological tools like the Geometric Sketch Pad have shown great value in helping students grasp geometric properties by enabling them to create dynamic structures. Similarly, GeoGebra has effectively enhanced students' geometric thinking, especially in achieving Van Hiele's levels 3 and 4 (Tutkun & Ozturk, 2013; Susan Ansah, Asiedu-Addo & Teye Kabutey, 2022). However, the application of GeoGebra has been demonstrably practical in substantially enhancing understanding across both primary and tertiary education levels (Kutluca, 2013), particularly concerning issues related to the Pythagorean theorem. Research suggests that students with high academic achievement can attain all four levels of Van Hiele's framework when instructed using GeoGebra (Wulandari et al., 2021). Henceforth, Atteh (2020) highlights the importance of using structured, learner-centred instructional sequences in geometry.

This study explores how middle-stage students understand Pythagoras' Property and right-angled triangles through Van Hiele's theory and uses GeoGebra as a supporting tool. Pythagoras, a Greek philosopher, discovered a fundamental property of right-angled triangles, which was later formalized as the Pythagorean Theorem. This theorem plays an important role in many fields, underpinning the development of various tools and technologies that impact the modern world.

Van Hiele's Theory:

The study uses the Van Hiele theory of geometric thinking, developed by Pierre and Dina Van Hiele-Geldof in 1986. According to this theory, learners progress through five levels of geometric understanding arranged hierarchically. Visualization (learners observe and identify geometric shapes by their appearance), Analysis (learners understand shape properties but may struggle to explain how these properties relate to each other), Abstraction (learners draw logical conclusions about geometric shapes and organize their understanding), Deduction (learners apply deductive reasoning to connect general principles with specific examples), and Rigor (learners grasp formal logical deduction and are able to develop precise geometric proofs for comparison and verification).

This study aims to assess the level of middle-stage students' understanding of the Pythagorean theorem in this context, using GeoGebra to help them progress through Van Hiele's levels of geometric understanding.

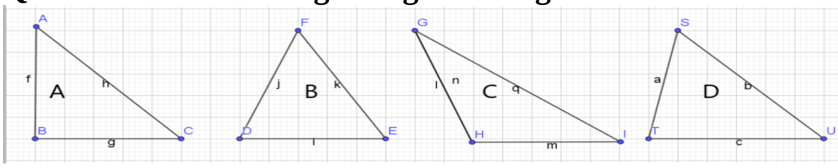
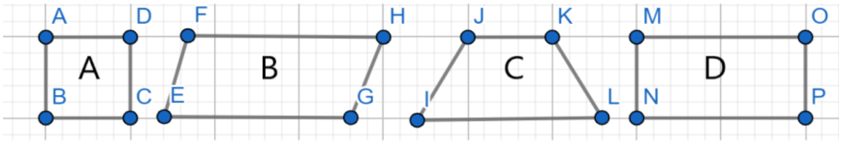
The Objective of the Study:

1. To assess the understanding level of middle-stage students of the Pythagorean Theorem using Van Hiele's Theory.

Methodology:

The qualitative research was conducted in Darbhanga, Bihar, at a randomly chosen Government middle school in Garri Village, Jale block. From sections A and B, 18 eighth-grade students were randomly chosen based on their academic performance. A total of six students were randomly selected from high, average, and below-average groups. A paper-and-pencil test was constructed applying Van Hiele's levels of geometric understanding with expert consultation and administered. The test included multiple-choice and open-ended questions, with GeoGebra instruction. To assess understanding of the Pythagorean theorem, one-on-one face-to-face interviews with unstructured questions were conducted, each lasting five minutes, totaling about 100 minutes for the entire process. This method explored students' progression from visualisation to deduction in understanding the theorem using traditional testing and GeoGebra-supported instruction.

Table 1:
Pythagorean Theorem
According to Van Hiele's Theory, Level

Levels	Questions
0 (Visualization)	**Q: Which one is the Right-angled triangle?**  **Q: Which one is the Square?**  **Q: Draw more Right-angled triangles & squares in your notebook.**
1 (Analysis)	**Q: Measure the length and the angle of the triangle and the square with the help of a ruler and a protractor.** **Q: Write the measured lengths and angles on the sides of the triangle & square.**
2 (Informal Deduction)	**Q: Does your triangle fulfill the properties of being a right-angled triangle?** **Q: If yes, then construct squares on each edge.**
3 (Deduction)	**Q: Compute the area of the square that is constructed on the edges of a right-angled triangle?** **Q: Does the square on the hypotenuse equal the sum of the squares on the legs?** **Q: Is it fulfilling the Pythagorean property, i.e., $A^2=B^2+C^2$** **Q: Prove it with another drawn triangle.**

***(GeoGebra assisted Instruction was provided by the researcher at each level in solving the problems)**

Example:

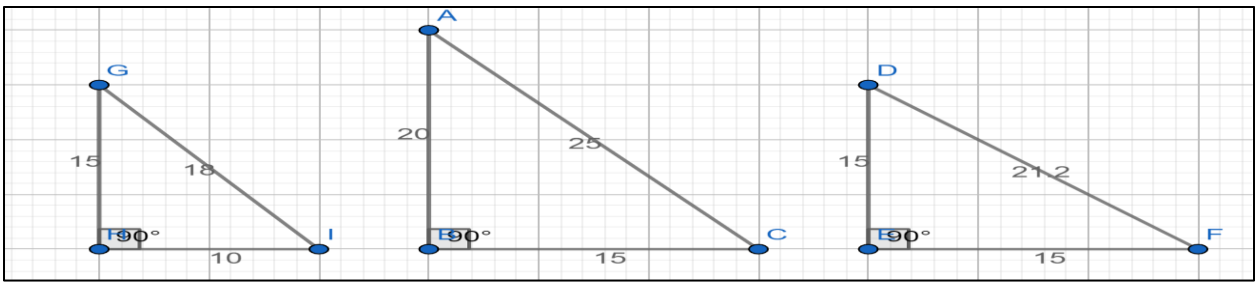


Figure 1: Right-angled triangles with lengths and angles on GeoGebra

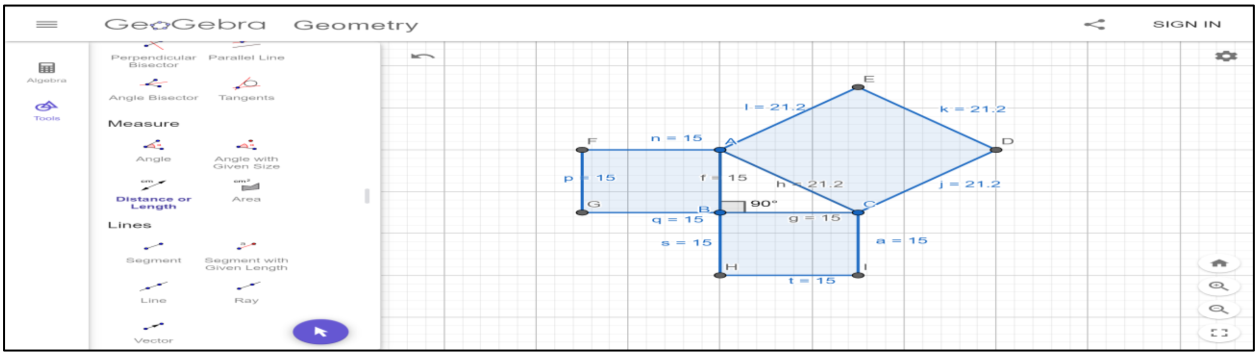


Figure2: Right-angled triangles and squares with lengths and angles on GeoGebra

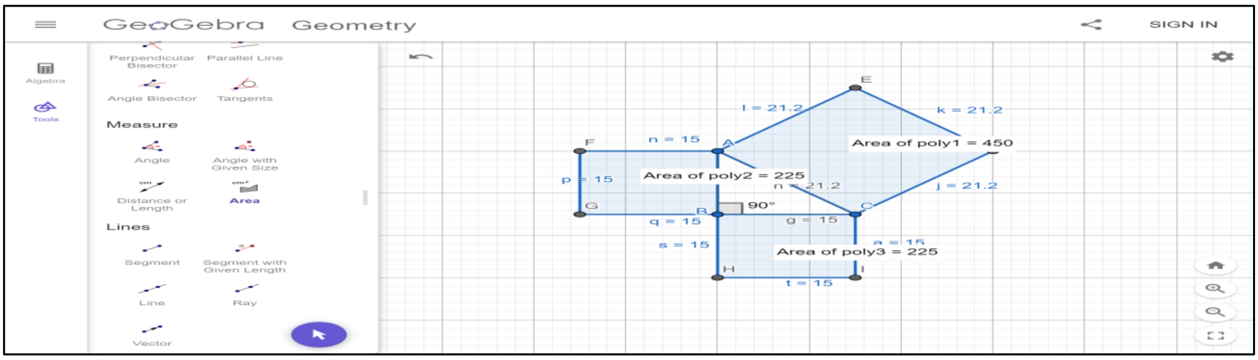


Figure 3:The square on the hypotenuse equals the sum of the squares on the legs.

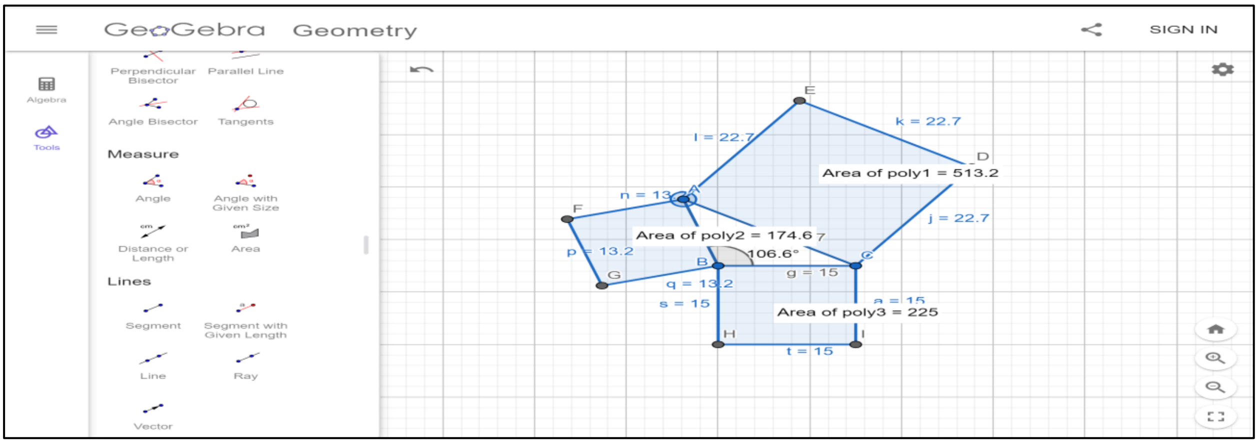


Figure 4: If the triangle is not right-angled, then the hypotenuse's square does not equal the sum of the squares of the legs.

Results and Discussion:

Data from paper-and-pencil tests and interviews were analysed through discourse analysis. Based on the test results on "If the Pythagorean Property Holds, the Triangle Must be Right-Angled" and interviews with the eighteen subjects, Van Hiele's geometric understanding level is analyzed.

High Performer Students:

The six high-performing students (S1, S2, S3, S4, S5, and S6) showcased their impressive understanding of right-angled triangles and the Pythagorean theorem. They skillfully identified right-angled triangles and squares during the paper-and-pencil test. Additionally, they happily drew triangles and squares of various sizes in their notebooks, clearly showing their strong understanding of geometric visualization.

At the analytical level, the students precisely measured the sides of the triangles and squares, accurately documenting the lengths and marking the edges. Transitioning to the informal deduction phase, they offered explicit justifications for the right-angled nature of the specified triangles and subsequently constructed squares on the triangles' sides. They computed the areas of these squares, enhancing their understanding of the correlation between the sides and the Pythagorean theorem.

The students demonstrated that the square on the hypotenuse equals the sum of the other two sides' squares, reaching Van Hiele's level 4 (Deduction). They illustrated that if the Pythagorean theorem is satisfied, the triangle is right-angled, substantiating their conclusions with multiple examples. Using GeoGebra at every stage augmented their comprehension and adeptly directed them through each degree of the assessment.

In the follow-up interviews, they were asked to draw a right-angled triangle with equal sides. S1, S3, and S5 attempted the drawing but concluded that it was impossible, correctly recognising the geometric limitation. Similarly, S2, S4, and S6 agreed that creating a right-angled triangle with equal sides was not feasible.

When questioned about right-angled triangles with two 90-degree angles, all students confidently asserted that such a triangle can only contain one 90-degree angle. They also clarified that the side opposite the 90-degree angle is the hypotenuse, which is always the longest side of the triangle. These answers demonstrate the students' strong grasp of geometric principles, their capacity to reason logically, and their thorough understanding of the Pythagorean theorem.

Average Performer Students:

The six students exhibiting average performance (S7, S8, S9, S10, S11, and S12) displayed a fundamental, however pragmatic understanding of right-angled triangles and the Pythagorean theorem. All students accurately recognised the right-angled triangle and square in the multiple-choice questions and adeptly illustrated triangles and squares of varying dimensions in their notebooks.

During the analytical phase, participants demonstrated proficiency in measuring the lengths of the triangle sides and squares, precisely annotating the edges in their notebooks. Nonetheless, in the context of informal deduction, while they were able to construct squares on the triangle's corners, they failed to effectively illustrate the rationale for the triangle being right-angled. In the follow-up interview, their justifications were considered satisfactory but lacked depth.

At the deduction level, S7, S8, and S9 attempted to calculate the area of the squares on the edges, although S10, S11, and S12 struggled with this task. Furthermore, they were unable to demonstrate or justify the Pythagorean property (i.e., $A^2 = B^2 + C^2$) for the right-angled triangle and did not try to apply the property to other triangles. GeoGebra-based instruction was crucial at each stage to help them progress and fully understand the concepts, but they needed more time and clarification to grasp everything.

During the interviews, students hesitated when answering questions about right-angled triangles with equal sides, indicating they needed more clarification.

They correctly stated that a right triangle has only one 90-degree angle and confidently identified the hypotenuse, the side opposite the right angle, as always being the longest. At the same time, they demonstrated a basic understanding of right-angled triangle properties; their knowledge of more advanced geometric concepts was still evolving, highlighting the need for additional support and instruction to deepen their understanding.

Below Average Performer Students:

The six below-average performing students (S13, S14, S15, S16, S17, and S18) took initial steps in recognising right-angled triangles and squares during the visualisation stage. S13, S14, S15, S16, S17, and S18 correctly identified the shapes, while S14, S16, and S18 managed to draw various examples of right-angled triangles and squares in their notebooks. However, S15 struggled to produce further examples of these shapes, which hindered their progress during the test.

In the interviews, S13, S14, and S18 provided different examples of right-angled triangles and squares and their properties, indicating a partial understanding. However, the analysis stage proved more challenging. S16 attempted to measure the lengths of the triangle and square accurately, noting the dimensions on the edges, but S15 needed further clarification to complete this task. None of the students attempted the remaining questions at higher Van Hiele levels. However, during the interview, the students answered only a few questions, and their responses were often incomplete or unclear. They found it difficult to fully understand the concepts, as shown by their reluctance to explain their reasoning thoroughly. Although GeoGebra instruction was introduced at each stage, these students needed additional support and time to grasp the material.

Therefore, these students solved the first three problems successfully, showing they had reached Van Hiele's Level 1 (Visualisation). However, their understanding remained at a basic level, i.e., Visualisation, requiring more specific guidance to progress to higher levels of geometric understanding.

Conclusion:

The study finds that high-achieving students effectively progressed through all four stages of Van Hiele's Theory in comprehending the Pythagorean theorem. GeoGebra's pedagogical assistance facilitated their seamless progression across levels, enhancing their capacity to perceive and reason through geometric concepts. Furthermore, GeoGebra-assisted instruction enabled pupils with average performance to attain the informal deduction level of Van Hiele's theory. Nonetheless, many had difficulties following the instructions or needed an alternative tempo to comprehend the topics successfully. These pupils made progress but required further help to attain a deeper understanding. Conversely, below-average children attained just the visualisation level when they could identify geometric shapes but struggled to advance further. Despite undergoing GeoGebra-based training, some students encountered difficulties adhering to the procedural stages and needed further organized coaching to enhance their understanding.

The results indicate that educators should emphasise conceptual learning rather than rote memorisation to comprehend geometric principles better. Computer-aided instruction should be incorporated into geometry education, particularly utilising technologies such as GeoGebra, to enhance students' engagement with subjects. Educators can assist students in linking geometric properties to real-world applications by offering various ICT-based examples, thereby enriching their learning experience, and promoting a more profound understanding of geometry.

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