International Journal of Advance and Applied Research
www.ijaar.co.in
ISSN – 2347-7075 Impact Factor – 8.141
Peer Reviewed Bi-Monthly
Vol. 6 No. 18 March - April - 2025

Alternative Proofs For Some Simple Inequalities Involving Inverse Trigonometric and Inverse Hyperbolic Functions

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DOI - 10.5281/zenodo.15259393

Abstract:
In this article, we have provided simple alternative proofs to the existing inequalities involving inverse trigonometric and inverse hyperbolic functions by using series expansions of these functions. Also, some required lemmas and their proofs have been encoporated.

Keywords: Inverse Trigonometric Function, Inverse Hyperbolic Function, Alternative Proofs, Simple Inequalities.

MSC(2010): 26D05, 26D07, 26D20, 33B10.

Introduction:

Inverse trigonometric functions and inverse hyperbolic functions have vide variety of applications in the various branches and sub-branches of science which directly and indirectly affects human life. Hence theses functions have great importance and needed to be studied extensively. Many mathematicians have studied various versions of inequalities involving these functions. Here, we write some inequalities of our interest studied by [1, 2] and some required lemmas as follows.

Theorem 1[*1, Proposition 1****]***

If $\theta \in (0,1)$, then we have

$$1+\frac{\theta^2}{6}<\frac{\sin^{-1}\theta}{\theta}<1+\left(\frac{\pi-2}{2}\right)\theta^2, \qquad (1)$$

where $\{6^{-1}, 2^{-1}(\pi-2)\}$ is set of best possible constants to hold the inequality.

Lemma 1

For any natural number k, the inequality $\frac{1}{3}<\frac{(2k+1)^2}{(k+1)(2k+3)}<2$ is true. (2)

Lemma 2

For any whole number k, the inequality $\left(\frac{2}{3}\right)^k<\frac{(2k)!}{(k!)^2(2k+1)}$ is true. (3)

Theorem 2[*1, Proposition 2****]***

If $\theta \in (0,1)$, then we have

$$\frac{6}{6-\theta^2}<\frac{\sin^{-1}\theta}{\theta}<\frac{\pi}{\pi-(\pi-2)\theta^2}, \qquad (4)$$

where $\{6^{-1}, \pi^{-1}(\pi-2)\}$ is set of best possible constants to hold the inequality.

Theorem 3[*2,Theorem 2.3****]***

Let $\theta \in (0,1)$.

Then $\frac{3}{3-\theta^2}<\frac{\tanh^{-1}\theta}{\theta}<\frac{1}{1-\theta^2}$, (5)

where $\{3^{-1}, 1\}$ is the set of best possible constants.

In this article, we have provided alternative proofs to the existing inequalities (1), (4), (5) involving inverse trigonometric and inverse hyperbolic functions by using series expansions of these functions. Proofs of some lemmas are also provided as an aid. Power series expansions of $\sin^{-1}\theta$ [3,1.641] and $\tanh^{-1}\theta$ [3,1.643] are as follows:

$$\frac{\sin^{-1}\theta}{\theta} = \sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}\theta^{2k}$$
$$= \sum_{k=0}^{\infty} a_k\theta^{2k}, \theta < 1 \qquad (6)$$

$$\frac{\tanh^{-1}\theta}{\theta} = \sum_{k=0}^{\infty}\frac{1}{2k+1}\theta^{2k}, \theta < 1, \qquad (7)$$

Proof of Theorem 1:

Let us set $\psi(\theta) = \theta^{-2}\left(\frac{\sin^{-1}\theta}{\theta} - 1\right)$

Using (6) in above,

we get $\psi(\theta) = \sum_{k=1}^{\infty} a_k\theta^{2k-2}$ and obviously $a_k > 0, \forall\, k \in \mathbb{N}$.

$\therefore \psi'(\theta) > 0, \forall\, \theta < 1$.

Implies that $\psi(\theta)$ is an increasing function on $(0,1)$. One can easily find the limits at the end points, $\psi(0+) = 1/6$ and $\psi(-1) = (\pi-2)/2$.

QED

Proof of Lemma 1:

For a natural number k,

inequality $\frac{1}{3} < \frac{(2k+1)^2}{(k+1)(2k+3)} < 2$ is true

$\Leftrightarrow\ (k+1)(2k+3) < 3(2k+1)^2$

and $(2k+1)^2 < 2(k+1)(2k+3)$

$\Leftrightarrow 0 < 3(4k^2+4k+1) - (2k^2+5k+3)$

and $0 < 2(2k^2+5k+3) - (4k^2+4k+1)$

$\Leftrightarrow\ 0 < k(10k+7), \forall\, k \in \mathbb{N}$ and

$0 < 6k+5,\ \forall\, k \in \mathbb{N}$.

Hence the proof of lemma.

QED

Proof of Lemma 2:

For a whole number k, we will prove inequality $\left(\frac{2}{3}\right)^k < \frac{(2k)!}{(k!)^2(2k+1)}$, it is equivalent to the inequality $\frac{1}{6^k} < \frac{(2k)!}{2^{2k}(k!)^2(2k+1)}$.

Now let us set, a function of k as

$P(k) := \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} - \frac{1}{6^k}$.

To prove the inequality, we must show that $P(k) \geq 0, k = 0,1,2,\ldots$ We will prove this by the principle of mathematicle induction.

For $k = 0,\ P(k) = 0 \geq 0$,

For $k = 1,\ P(k) = \frac{1}{6} - \frac{1}{6} = 0 \geq 0$,

For $k = 2,\ P(k) = \frac{3}{40} - \frac{1}{36} > 0$,

For $k = 3,\ P(k) = \frac{5}{112} - \frac{1}{216} > 0$.

Thus, $P(k) \geq 0, for\ k = 0,1,2,3$

Induction: Assume that $P(k) \geq 0$ for $k = m$

i.e. $P(m) = \frac{(2m)!}{2^{2m}(m!)^2(2m+1)} - \frac{1}{6^m} \geq 0$.

Now consider the following,

$P(m+1)$

$= \frac{(2m+2)!}{2^{2m+2}((m+1)!)^2\,(2m+3)} - \frac{1}{6^{m+1}},$

$= \frac{(2m+1)^2}{2(m+1)(2m+3)} \times \frac{(2m)!}{2^{2m}(m!)^2(2m+1)} - \frac{1}{6^m} \times \frac{1}{6},$

$\geq \frac{1}{6}\left(\frac{(2m)!}{2^{2m}(m!)^2(2m+1)} - \frac{1}{6^m}\right)$, since *Lemma 1*.

≥ 0, since $P(m) \geq 0$.

Thus, $P(m) \geq 0 \Rightarrow P(m+1) \geq 0$.

$\therefore$ By the principle of mathematical induction $P(k) \geq 0, k = 0,1,2,\ldots$

Hence the inequality (3) is proved.

QED

Proof of Theorem 2:

Using *Lemma 2* in (6), we obtained

$$\frac{\sin^{-1}\theta}{\theta} = \sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}\theta^{2k}$$
$$\geq \frac{\sin^{-1}\theta}{\theta} = \sum_{k=0}^{\infty}\frac{\theta^{2k}}{6^k} = \sum_{k=0}^{\infty}\left(\frac{\theta^2}{6}\right)^k$$
$$= \frac{1}{1-\frac{\theta^2}{6}} = \frac{6}{6-\theta^2}.$$

Thus the left inequality is proved.

QED

Proof of Theorem 3:

For $\theta < 1$, we have (7),

$$\frac{\tanh^{-1}\theta}{\theta} = \sum_{k=0}^{\infty}\frac{1}{2k+1}\theta^{2k}$$

$$< \sum_{k=0}^{\infty}\theta^{2k} = \frac{1}{1-\theta^2},\ since\ \frac{1}{2k+1} < 1$$

And we have a very obvious inequality $1 < 2k+1 < 3^k \Leftrightarrow 1 > \frac{1}{2k+1} > \frac{1}{3^k}$, which can be utilised in the following

$$\frac{\tanh^{-1}\theta}{\theta} = \sum_{k=0}^{\infty}\frac{1}{2k+1}\theta^{2k}$$

$$> \sum_{k=0}^{\infty}\frac{1}{3^k}\theta^{2k}$$

$$= \sum_{k=0}^{\infty}\left(\frac{\theta^2}{3}\right)^k = \frac{1}{1-\frac{\theta^2}{3}} = \frac{3}{3-\theta^2}$$

Thus by combinig above, we get

$$\frac{3}{3-\theta^2} < \frac{\tanh^{-1}\theta}{\theta} < \frac{1}{1-\theta^2}.$$

QED

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