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Original Article

Uniqueness Results on Linear Differential Polynomials and Differential-Difference Polynomials of Meromorphic Function

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Abstract:

In this paper we have shown that, a finite order transcendental meromorphic function $f(z)$ in the complex plane can not share the Borel exceptional value β CM with a differential-difference polynomial. And we have also proved that, if a hyper order strictly less than 1 function $f(z)$ and a linear differential polynomial $L(f)$ share a and ∞ CM then $f(z) \equiv L(f)$.

Math. Subj. Classification 2020: *30D20, 30D30 30D35.*

Key Words and Phrases: Meromorphic function, linear, differential-difference polynomial, sharing, exceptional value.

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Introduction:

In the classical uniqueness theorems (e.g., the shared value condition), two meromorphic functions were required to share all values (with specified multiplicities) in order to guarantee uniqueness. In contrast, these subsequent works relaxed this assumption by considering weaker versions of shared values or different types of shared value conditions. In the study of uniqueness problems, a major advancement followed the initial Nevanlinna results, where researchers began exploring more general situations by relaxing the assumption that two meromorphic functions share exactly the same values. One key line of development involved considering various operators applied to the meromorphic function to study the uniqueness of these functions. The shift, derivative, and difference operators played pivotal roles in this exploration. By considering these operators and relaxing the traditional assumptions of exact value sharing, many authors have been able to derive stronger or more general uniqueness results (see Yang et al. [10], Heittokangas et al. [6] and [5], S. Chen [1], Chen [2], Li et al. [7], Zhang et al. [12]). They often aim to balance between generalizing the conditions of uniqueness while still keeping the results rigorous and useful for applications in complex analysis, especially in meromorphic function theory. We investigate how the behaviour of $f(z)$ and its linear differential, differential-difference polynomials affect uniqueness.

Basic Theorems:

Nevanlinna work in value distribution theory provides essential results in complex analysis, especially for understanding uniqueness problems in the context of meromorphic functions. The two classic results that often serve as the foundation are:

Theorem 2.1. [9] If five distinct values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C} \cup \infty$ IM are shared by two meromorphic functions $f(z)$ and $g(z)$ then $f(z) = g(z)$.

Theorem 2.2. [9] Suppose, $f(z)$ and $g(z)$ be two meromorphic functions sharing four distinct values $a_1, a_2, a_3, a_4 \in \mathbb{C} \cup \infty$ CM then $f(z) = g(z)$ or $f(z)$ is a Mobius transformation of $g(z)$.

These theorems have profound implications for uniqueness theory. And they provide a framework for determining when two meromorphic functions must be identical, given the value distribution information.

In 2017, F. Lu-W. Lu [8] proved the following for three distinct values.

Theorem 2.3. [8] Let f be a transcendental meromorphic function of finite order, and let $f = f(z + c) - f(z)(\neq 0)$. If f and $\Delta(f)$ share three distinct values e_1, e_2, ∞ CM, then $f = \Delta(f)$.

In 2019, Gao et al.,[3] obtained the following result for $f(z)$ sharing three distinct periodic functions with its higher order difference operator.

Theorem 2.4. [3] Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1 such that $\Delta^n f(z) \neq 0$. If $f(z)$ and $\Delta^n f(z)$ share three distinct periodic functions $a, b, c \in S(f) \cup \infty$ with period 1 CM, then $f(z) \equiv \Delta^n f(z)$.

In 2021, X. Qi and L. Yang [9] considered the sharing value as 2CM to obtain the below result.

Theorem 2.5. [9] Let $f(z)$ be a transcendental meromorphic function of hyper order strictly less than 1 and let $a(\neq 0) \in S(f)$. If $f(z)$ and $\Delta^n f(\neq 0)$ share a and ∞ CM and $2\delta(0, f) + 2\Theta(\infty, f) > 3$ then $f(z) \equiv \Delta^n(f)$.

In 2013, Chen and Yi [2] proved the following result for the uniqueness of entire function and its difference operator.

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Theorem 2.6. [2] Let f be a finite order transcendental entire function which has a finite Borel exceptional value a, and let $c \in \mathbb{C}$ be a constant such that $f(z) \neq f(z+c)$. If $\Delta_c f(z)$ and $f(z)$ share the value a CM, then $a = 0$ and

$$\frac{\Delta_c f(z)}{f(z)} = \lambda,$$

where λ is some nonzero constant.

In 2021, X. Zhang and X. Chen [13] extended the result for meromorphic function and difference polynomial to prove the following theorem.

Theorem 2.7. [13] Let f be a transcendental meromorphic function of finite order. Suppose that $\beta \in \mathbb{C}$ and ∞ are Borel exceptional values of $f, P(z,f) = a_n f(z+nc) + a_{n-1} f(z+(n-1)c) + \cdots + a_1 f(z+c) + a_0 f(z)\ (\neq 0)$ be a difference polynomial of f, where $a_k (k = 0, 1, 2, \cdots, n)$ are not all zero complex numbers. If $\beta \neq 0$ then $P(z,f)$ and f can not share the value β CM.

Lemmas:

Lemma 3.1. **[13]** Let f be a transcendental meromorphic function of finite order. Suppose that $\beta \in \mathbb{C}$ and ∞ are Borel exceptional values of f, then

$$f(z) = A(z) e^{P(z)} + \beta,$$

where $P(z)$ is a polynomial and $A(z)$ is a meromorphic function such that $\lambda(A) = \lambda(\beta, f)$, $\lambda\left(\frac{1}{A}\right) = \lambda\left(\frac{1}{f}\right)$ and

$$\sigma(A) \leq max\{\lambda(\beta,f), \lambda\left(\frac{1}{f}\right)\} < \sigma(f) = degP(z).$$

Lemma 3.2. [11] Let $f_1(z)$ and $f_2(z)$ be non-constant meromorphic functions in the complex plane and c_1, c_2, c_3 be non-zero constants. If $c_1 f_1 + c_2 f_2 = c_3$, then

$$T(r,f_1) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r,f_1) + S(r,f_1).$$

Lemma 3.3. [13] Let $f(z)$ be a non-constant meromorphic function of hyper-order strictly less than 1. Then

$$N\left(r, \frac{1}{\Delta^n f}\right) \leq T(r, \Delta^n f) - T(r,f) + N\left(r, \frac{1}{f}\right) + S(r,f).$$

Lemma 3.4. [13] Let $f(z)$ be a non-constant meromorphic function in the complex plane, and let $a(\neq 0) \in S(f)$. Set

$$H = \frac{\Delta^n f - a}{f - a}.$$

If f and $\Delta^n f(\neq 0)$ share a and ∞ CM, then

$$N\left(r, \frac{1}{H}\right) = S(r,f), \qquad N(r, H) = S(r,f).$$

Lemma 3.5. [9] Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1, then we have

$$T(r, f(z+c)) = T(r,f) + S(r,f).$$

Lemma 3.6. [4] Let $f(z)$ be a meromorphic function of hyper-order strictly less than 1. Then,

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r,f)$$

and

$$m\left(r, \frac{\Delta_c^n f}{f-a}\right) = S(r,f)$$

where a is a constant.

Lemma 3.7. [10] Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$ are linearly independent meromorphic functions satisfying the following identity

$$\sum_{j=1}^{n} f_j(z) = 1.$$

Then, for $1 \leq j \leq n$, we have

$$T(r, f_j) \leq \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r,D) - \sum_{k=1}^{n} N(r, f_k) - N\left(r, \frac{1}{D}\right) + S(r),$$

where D is the Wronskian determinant $W(f_1, f_2, \cdots, f_n)$,

$$S(r) = o(T(r)), \ (r \to \infty, r \notin E)$$

and

$$T(r) = max_{1 \leq k \leq n}\{T(r, f_k)\}$$

E is a set with finite linear measure.

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Main Results:

Definition 1 $L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_1 f = \sum_{i=0}^{k} a_i f^{(i)}$, where a_i, $i = 0, 1, \cdots, k$ are the complex constants and $a_k \neq 0$.

Remark 4.1 The following results are proved by using the similar methodology as in [[13, 9]]

Corollary 1. Let f be a transcendental meromorphic function of finite order. Suppose that $\beta \in \mathbb{C}$ and ∞ are BeV of f, $\phi(f) = \sum_{j\in J} A_j(z) f^{(k_j)}(z + a_j)$ be a differential-difference polynomial and $\phi(f) \neq 0$. If $\beta \neq 0$ then $\phi(f)$ and f can not share the value β CM.

Proof: Suppose that $\phi(f)$ and f share the value β CM, then

$$\frac{\phi(f) - \beta}{f(z) - \beta} = e^{h(z)}, \qquad (4.1)$$

where $h(z)$ is a polynomial, since β and ∞ are B.e.V of f then by Lemma (3.1), $f(z)$ can be written as

$$f(z) = A(z) e^{P(z)} + \beta, \qquad (4.2)$$

where $A(z)$ is a meromorphic function such that

$$\sigma(A) \leq \max\left\{\lambda(\beta, f), \lambda\left(\frac{1}{f}\right)\right\} < \sigma(f) = degP(z).$$

It follows from (3.1) and (3.2) that

$$\frac{\sum_{j\in J} A_j(z) T_j(z+a_j) e^{P(z+a_j)} - \beta}{A(z) e^{P(z)}} = e^{h(z)}. \qquad (4.3)$$

That is,

$$\frac{\sum_{j\in J} A_j(z) T_j(z+a_j) e^{P(z+a_j) - P(z)}}{A(z)} - \frac{\beta}{A(z)} e^{-P(z)} = e^{h(z)}. \quad (4.4)$$

As $\sigma(A) < degP(z)$ and $deg(P(z + aj) - P(z)) \leq (degP(z)) - 1 = \sigma(f) - 1$, $i = 0, 1, 2, \cdots, n$ then $\frac{\sum_{j\in J} A_j(z) T_j(z+a_j)}{A(z)} e^{P(z+a_j) - P(z)}$ is a small meromorphic function respective to $\frac{\beta}{A(z)} e^{-P(z)}$.

Applying the second fundamental theorem to $\frac{\beta}{A(z)} e^{-P(z)}$, we know that

$$\lambda\left(\frac{\sum_{j\in J} A_j(z) T_j(z+a_j)}{A(z)} e^{P(z+a_j) - P(z)} - \frac{\beta}{A(z)} e^{-P(z)}\right) = degP(z).$$

This contradicts with $e^{h(z)} \neq 0$. Thus $f(z)$ and $\phi(z)$ cannot share the value β CM. ■

Remark 4.2 The corollary shows that either $\phi(f)$ and $f(z)$ share the value $a \neq \beta$ for any $\beta \in \mathbb{C}$ or share 0 CM if $\beta = 0$.

Theorem 4.1. Let $f(z)$ be a transcendental meromorphic function of hyper order strictly less than 1 and let $a(\neq 0) \in S(f)$. If $f(z)$ and $L(f)$ share a and ∞ CM and $2\delta(0, f) + 2\Theta(\infty, f) > 3$ then $f(z) \equiv L(f)$.

Proof: Suppose that $f \neq L(f)$. Set

$$H = \frac{L(f) - a}{f - a}. \qquad (4.5)$$

Case 1. If $H(z)$ is a constant, i.e. $H(z) = A$. Then by the assumption that $f \neq L(f)$ and the value sharing assumption, we have $A \neq 1$ and $A \neq 0$. Hence (4.5) can be rewritten as,

$$\frac{L(f)}{a} - \frac{Af}{a} = 1 - A.$$

This together with Lemmas (3.2), (3.3) and the value sharing assumption, it follows that

$$T(r, L(f)) \leq T\left(r, \frac{L(f)}{a}\right) + S(r, f),$$
$$\leq \overline{N}\left(r, \frac{a}{L(f)}\right) + \overline{N}\left(r, \frac{a}{Af}\right) + \overline{N}\left(r, \frac{L(f)}{a}\right) + S(r, f),$$
$$\leq T(r, L(f)) - T(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r, f).$$

That is, $T(r, f) \leq 2N\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r, f)$, contradicts the assumption that $2\delta(0, f) + 2\Theta(\infty, f) > 3$.

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Case 2. If $H(z)$ is not a constant, then we can write (4.5) as in the following form

$$\frac{L(f)}{a} - \frac{Hf}{a} + H = 1. \qquad (4.6)$$

Subcase 2.1 Suppose that $\frac{L(f)}{a}, \frac{Hf}{a}, H$ are linearly independent meromorphic functions. Then, by Lemma (3.7) we have

$$T(r, L(f)) \le T\left(r, \frac{L(f)}{a}\right) + S(r,f),$$

$$\le N\left(r, \frac{a}{L(f)}\right) + N\left(r, \frac{a}{Hf}\right) + N\left(r, \frac{1}{H}\right) + N(r, D)$$

$$- N\left(r, \frac{Hf}{a}\right) - N(r, H) + S(r), \ (r \to \infty, r/\in E), \qquad (4.7)$$

where

$$D = \begin{vmatrix} \frac{L(f)}{a} & -\frac{Hf}{a} & H \\ \left(\frac{L(f)}{a}\right)' & \left(-\frac{Hf}{a}\right)' & H' \\ \left(\frac{L(f)}{a}\right)'' & \left(-\frac{Hf}{a}\right)'' & H'' \end{vmatrix} \qquad (4.8)$$

On the other hand, from (4.5) and Lemma (3.5), we see

$$T(r, H) \le T(r, f) + T(r, L(f)) + S(r, f) \le (1 + a^k) T(r, f) + S(r, f).$$

Therefore, $S(r) = S(r, f)$. On the other hand, from (4.6) and (4.8), we conclude that

$$D = \begin{vmatrix} 1 & -\frac{Hf}{a} & H \\ 0 & \left(-\frac{Hf}{a}\right)' & H' \\ 0 & \left(-\frac{Hf}{a}\right)'' & H'' \end{vmatrix}$$

Hence,

$$N(r, D) = N\left(r, \left(\frac{Hf}{a}\right)' H'' - \left(\frac{Hf}{a}\right)'' H'\right) \le N\left(r, \left(\frac{Hf}{a}\right)''\right) + N(r, H'').$$

Further we have

$$N(r, D) - N\left(r, \frac{Hf}{a}\right) \le 2\bar{N}\left(r, \frac{Hf}{a}\right) + N(r, H''). \qquad (4.9)$$

Thus from (4.7), (4.9) and Lemma (3.4), it follows that

$$T(r, L(f)) \le N\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{f}\right) + 2\bar{N}(r, f) + S(r, f).$$

This together with Lemma (2.3), we have

$$T(r, L(f)) \le T(r, L(f)) - T(r, f) + 2N\left(r, \frac{1}{f}\right) + 2\bar{N}(r, f) + S(r, f).$$

Which implies that

$$T(r, f) \le 2N\left(r, \frac{1}{f}\right) + 2\bar{N}(r, f) + S(r, f),$$

contradicts the assumption $2\delta(0, f) + 2\Theta(\infty, f) > 3$.

Subcase2.2 Suppose that $\frac{L(f)}{a}, \frac{Hf}{a}, H$ are linearly dependent meromorphic functions. Then, there exist three numbers B, C, D that are not all zero such that

$$B\frac{L(f)}{a} - C\frac{Hf}{a} + DH = 0. \qquad (4.10)$$

Firstly, we confirm that $B \ne 0$, otherwise, if $B = 0$, then we have $CD \ne 0$. Rewrite

$$H\left(\frac{Cf}{a} - D\right) = 0.$$

Since $H \ne 0$, we have $f = \frac{aD}{C}$, which is a contradiction and so $B \ne 0$. Combining (4.6) and (4.10),

$$(C - B)\frac{Hf}{a} + (B - D)H = B. \qquad (4.11)$$

If $C - B \ne 0$, $B - D \ne 0$. then (4.11) can be rewritten as,

$$\frac{C-B}{B}\frac{f}{a} - \frac{1}{H} = \frac{D-B}{B}. \quad (4.12)$$

Applying Lemmas (3.2) and (3.4) into (4.12) we have

$$T(r, f) = T\left(r, \frac{f}{a}\right) + S(r, f),$$

$$\le \bar{N}\left(r, \frac{a}{f}\right) + \bar{N}(r, H) + \bar{N}\left(r, \frac{f}{a}\right) + S(r, f),$$

$$\le \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}(r, f) + S(r, f),$$

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contradicts the assumption that $2\delta(0,f)+2\Theta(\infty,f)>3$. If $C-B=0$, $B-D\neq 0$. then by (4.11) we get $H=\frac{B}{B-D}$ which contradicts the assumption of case 2.

If $C-B\neq 0$, $B-D=0$. then from (4.11) we have

$$\frac{Hf}{a}=\frac{B}{C-B}. \qquad (4.13)$$

combining (4.6) and (4.13) implies that

$$\frac{L(f)}{a}+H=\frac{C}{C-B}. \qquad (4.14)$$

If $C\neq 0$, then by Lemma $(3.2)-(3.4)$ and the value sharing assumption, we have

$$T(r,L(f)) \leq T\left(r,\frac{L(f)}{a}\right)+S(r,f),$$

$$\leq \bar{N}\left(r,\frac{a}{L(f)}\right)+\bar{N}\left(r,\frac{1}{H}\right)+\bar{N}\left(r,\frac{L(f)}{a}\right)+S(r,f),$$

$$\leq T(r,L(f))-T(r,f)+N\left(r,\frac{1}{f}\right)+\bar{N}(r,f)+S(r,f),$$

which means that $T(r,f)\leq N\left(r,\frac{1}{f}\right)+\bar{N}(r,f)+S(r,f)$,

contradicts the assumption $2\delta(0,f)+2\Theta(\infty,f)>3$.

If $C=0$, then from (4.14), we know that

$$L(f)=-aH. \qquad (4.15)$$

Moreover, by (4.6) we have

$$Hf=-a.$$

This together with (4.15), we get

$$fL(f)=a^2. \qquad (4.16)$$

On one hand, from the assumption that f and $L(f)$ share ∞ CM and (4.16), we obtain that

$$N\left(r,\frac{1}{f}\right)=S(r,f).$$

On the other hand, it follows by Lemma (3.6) and (4.16), that

$$m\left(r,\frac{1}{f}\right)=\frac{1}{2}m\left(r,\frac{1}{f^2}\right)+S(r,f),$$

$$\leq m\left(r,\frac{fL(f)}{f^2}\right)+m\left(r,\frac{1}{fL(f)}\right)+S(r,f),$$

$$\leq m\left(r,\frac{1}{a^2}\right)+S(r,f)=S(r,f).$$

Hence, $T(r,f)=S(r,f)$, which is impossible. Hence the proof. ■

References:

1. S. Chen. On uniqueness of meromorphic functions and their difference operators with partially shared values. *Computational Methods and Function Theory*, 18:529–536, 2018.
2. Z.-X. Chen and H.-X. Yi. On sharing values of meromorphic functions and their differences. *Results in Mathematics*, 63(1-2):557–565, 2013.
3. Z. Gao, R. Korhonen, J. Zhang, and Y. Zhang. Uniqueness of meromorphic functions sharing values with their nth order exact differences. *Analysis Mathematica*, 45:321– 334, 2019.
4. R. Halburd, R. Korhonen, and K. Tohge. Holomorphic curves with shiftinvariant hyperplane preimages. *Transactions of the American Mathematical Society*, 366(8):4267–4298, 2014.
5. J. Heittokangas, R. Korhonen, I. Laine, and J. Rieppo. Uniqueness of meromorphic functions sharing values with their shifts. *Complex Variables and Elliptic Equations*, 56(1-4):81–92, 2011.
6. J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, and J. Zhang. Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity. *Journal of Mathematical Analysis and Applications*, 355(1):352–363, 2009.
7. S. Li and Z. Gao. Entire functions sharing one or two finite values cm with their shifts or difference operators. *Archiv der Mathematik*, 97(5):475–483, 2011.
8. F. L¨u and W. L¨u. Meromorphic functions sharing three values with their difference

operators. *Computational Methods and Function Theory*, 17(3):395–403, 2017.

9. X. Qi and L. Yang. Uniqueness of meromorphic functions with their nth order differences. *Bulletin of the Iranian Mathematical Society*, 47(5):1491–1504, 2021.
10. C. Yang and H. Yi. Uniqueness theory of meromorphic functions, ser. *Mathematics and its Applications*. Dordrecht: Kluwer Academic Publishers Group, 557, 2003.
11. C.-C. Yang and H.-X. Yi. Uniqueness theory of meromorphic functions, volume 557. *Springer Science & Business Media*, 2004.
12. J. Zhang and L. Liao. Entire functions sharing some values with their difference operators. *Science China Mathematics*, 57:2143–2152, 2014.
13. X. Zhang and X. Chen. Uniqueness of difference polynomials. *AIMS Mathematics*, 6(10):10485–10494, 2021.