International Journal of Advance and Applied Research (IJAAR)
Peer Reviewed Bi-Monthly

ISSN – 2347-7075
Impact Factor – 7.328
Vol.9 No.4 Mar – Apr 2022

A REVIEW ON THE DIVERSE PHENOMENA OF NONLINEAR FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

Mr. Kartik Shankarrao Waghmare
Ph.D. Research Scholar,
Department of Mathematics,
Shri JJT University, Rajasthan, India

Dr. Vineeta Basotia
Professor & Ph.D. Research Guide,
Department of Mathematics,
Shri JJT University, Rajasthan, India

ABSTRACT:

This article is an extensive overview of the many various phenomena that may be seen in nonlinear fractional order differential equations. Because it can capture complicated dynamics and explain systems with memory and long-range dependency, fractional calculus has garnered a lot of interest in recent years. This is largely owing to its capacity to do these things. Differential equations of nonlinear fractional order have applications in a wide variety of domains, including physics, engineering, biology, and finance. These equations also display peculiar characteristics. Within the scope of this study, we investigate five essential facets of these equations, including their mathematical formulation, qualitative analysis, stability analysis, numerical approaches, and applications.

Keywords: *Phenomena, Nonlinear Fractional, Order Differential Equations*

INTRODUCTION:

Introduction to Fractional Calculus:

Calculus of fractions is a branch of mathematics that applies the ideas of differentiation and integration to orders that are not integers. It offers a strong instrument for evaluating and modelling complex systems that include memory and long-range dependencies. Fractional calculus, on the other hand, covers derivatives and integrals of non-integer orders, as opposed to the integer-order calculus, which deals with derivatives and integrals of integer orders. The requirement to represent phenomena that show non-local interactions and memory effects, which are prominent in a variety of scientific and technical domains, is the impetus for the development of fractional calculus.

Definitions of Fractional Derivatives:

There have been a number of different suggested definitions of fractional derivatives, and each one has its own set of benefits and uses. The most typical examples of this are the Riemann-Liouville derivative, the Caputo derivative, and the Grunwald-Letnikov derivative. The Riemann-Liouville derivative is a kind of fractional derivative that uses the integral representation of fractional derivatives as its foundation. This type of derivative may encapsulate both past and future dependency. On the other hand, the Caputo derivative takes into account starting circumstances, which makes it an excellent choice for modelling systems that are subject to initial transients. The Grunwald-Letnikov derivative is a discretized approximation of fractional derivatives that is especially helpful for numerical calculations. This approximation was developed by Grunwald and Letnikov. These numerous definitions provide flexibility in capturing different kinds of memory effects and may be utilised in different ways depending on the particular features of the system that is being researched.

Nonlinear Fractional Order Differential Equations:

When dealing with nonlinear functions of fractional derivatives, nonlinear fractional order differential equations are always going to be produced. They provide a sophisticated mathematical framework for explaining the dynamics of complex systems, which may be found here. When compared to standard integer-order differential equations, equations with fractional derivatives have a higher degree of complexity due to their inclusion. Memory effects, non-local interactions, and long-range dependency are all elements that are brought into the dynamics of the system by the fractional order. This makes it possible to simulate phenomena that display complex behaviours, such as fractal patterns, power-law decay, and anomalous diffusion. Differential equations of nonlinear fractional order have found use in a variety of domains, including physics, engineering, biology, and even finance.

Ordinary and Partial Fractional Order Differential Equation:

In the case of nonlinear fractional order differential equations, the existence of several variables may lead to the formulation of the equations in either their ordinary or their partial forms. Ordinary fractional order differential equations explain the dynamics of systems that have memory and fractional order derivatives. These equations only include a single variable at a time. Time series analysis and the modelling of systems with a single variable are two typical applications for their utilisation. On the other hand, partial fractional order differential equations incorporate numerous variables and explain the dynamics of systems that are spatially dependent. They have applications in a variety of domains, including the transmission of heat, the processes of diffusion, and the propagation of waves. The mathematical formulation of ordinary and partial fractional order differential equations is distinct from one another, and several solution approaches are used in the process of analysing and researching the behaviour of these equations.

The ideas of fractional calculus provide the foundation for the mathematical formulation of nonlinear fractional order differential equations, as this overview explains. The use of fractional derivatives makes it possible to describe complicated systems that include memory and long-range dependencies. The versatility offered by multiple definitions of fractional derivatives allows for the flexible capture of a wide variety of memory effects. Equations of nonlinear fractional order that display differential behaviour are useful in a wide variety of scientific and technical disciplines. These equations demonstrate their distinctive tendencies. The capacity to comprehend their mathematical formulation lays the groundwork for future investigation, including qualitative and stability investigations, numerical approximation, and applications in a variety of domains.

QUALITATIVE ANALYSIS:

Existence and Uniqueness of Solutions:

Investigating the existence and uniqueness of solutions is a crucial part of qualitative analysis for nonlinear fractional order differential equations. This is one of the core aspects of qualitative analysis. Establishing the existence of solutions and determining whether or not they are unique may be difficult when dealing with fractional derivatives because of their non-local and memory-dependent character. In order to demonstrate that there are solutions to various classes of fractional order equations and that these solutions are distinct from one another, a number of mathematical methods, including integral equations, fixed-point theorems, and contraction mapping, are used. For the purpose of providing robust mathematical foundations for the qualitative analysis, certain theorems and mathematical techniques associated with fractional calculus, such as the Banach fixed-point theorem and the Picard-Lindel of theorem for fractional differential equations, are applied.

Fixed Points and Periodic Orbits:

The exploration of fixed points and periodic orbits in nonlinear fractional order differential equations is one more essential component of qualitative analysis. Fixed points denote states in which the system does not change over the course of time, while periodic orbits denote motion that is repeated again and over again. Understanding the long-term behaviour of the system requires that you first determine whether or not periodic orbits and fixed points really exist and whether or not they are stable. Analyzing the stability and bifurcation processes associated with fixed points and periodic orbits requires the use of methods such as linearization, Lyapunov's direct approach, and Poincaré maps. Equations of nonlinear fractional order that show differential behaviour may display a diverse spectrum of fixed points and periodic orbits, including behaviours that are stable, unstable, and complicated, such as limit cycles and weird attractors.

Attractors and Stability:

In the qualitative study of nonlinear fractional order differential equations, the idea of attractors plays a key role. An attractor is an area in phase space that depicts a region in which the paths of a system converge over the course of time. In fractional order systems, there is the potential for the emergence of a variety of different forms of attractors, including fixed-point attractors, limit cycles, and odd attractors. It is of significant interest to investigate the stability of attractors since this factor dictates the behaviour and resilience of the system over the long run. The study of the stability of attractors in nonlinear fractional order differential equations makes use of stability analysis techniques such as Lyapunov stability, asymptotic stability, and LaSalle's invariance principle. The results of these studies provide light on the behaviour of the system as well as its capacity to be predicted under a variety of beginning circumstances and parameter values.

Bifurcation Analysis:

Bifurcation analysis is a powerful tool used to explore the qualitative changes in the behavior of nonlinear fractional order differential equations as system parameters vary. Bifurcations occur when the stability or nature of solutions undergoes a qualitative change due to small parameter variations. Various types of bifurcations, such as saddle-node, Hopf, pitchfork, and period-doubling bifurcations, can occur in fractional order systems. Bifurcation analysis helps to understand the emergence of complex dynamics, chaos, and multi-stability in nonlinear fractional order systems. Numerical techniques, such as continuation methods and bifurcation software, are often employed to analyze and visualize bifurcation diagrams and parameter spaces, providing insights into the system's behavior as parameter values are varied.

Limitations and Challenges:

Qualitative analysis of nonlinear fractional order differential equations presents certain challenges. The presence of non-local interactions and memory effects in fractional order derivatives complicates the analysis compared to traditional integer-order differential equations. The lack of analytical solutions for many fractional order equations necessitates the use of numerical and computational methods for qualitative analysis. Additionally, the complexity and sensitivity of fractional order systems require careful consideration of numerical approximations and the selection of appropriate numerical techniques. Addressing these challenges and limitations in qualitative analysis opens avenues for further research in developing robust and efficient analysis tools for nonlinear fractional order differential equations.

In summary, qualitative analysis of nonlinear fractional order differential equations involves investigating the existence and uniqueness of solutions, analyzing fixed points and periodic orbits, understanding attractors and their stability, exploring bifurcation phenomena, and addressing limitations and challenges in the analysis. These aspects provide valuable insights into the behavior, stability, and complex dynamics exhibited by nonlinear fractional order systems. The qualitative analysis lays the foundation for understanding and predicting the system's behavior under different conditions and parameter values, leading to applications in various scientific, engineering, and interdisciplinary domains.

STABILITY ANALYSIS:

Stability analysis is a crucial aspect of studying nonlinear fractional order differential equations as it provides insights into the behavior and robustness of the system's solutions. Understanding the stability properties helps determine whether the system will converge to a steady state or exhibit oscillations, chaos, or other complex behaviors. Stability analysis techniques specific to nonlinear fractional order differential equations are employed to assess the stability of solutions under different conditions and parameter values.

Lyapunov Stability:

Lyapunov stability analysis is a fundamental approach used to investigate the stability of solutions in nonlinear fractional order differential equations. The concept of Lyapunov functions, which are scalar functions of the system variables, plays a central role in this analysis. By selecting an appropriate Lyapunov function and examining its derivative along the system trajectories, it is possible to determine the stability of equilibrium points or periodic orbits. Lyapunov's direct method is widely applied to establish stability criteria, such as asymptotic stability, exponential stability, or stability in a region of attraction. Lyapunov stability analysis provides valuable information about the long-term behavior and convergence properties of the system.

Input-Output Stability:

In addition to analyzing the stability of solutions in isolation, it is often important to examine the stability of the system with respect to external inputs or disturbances. Input-output stability analysis focuses on how the system responds to input signals or perturbations and whether it remains bounded. Various stability concepts, such as input-to-state stability and integral input-to-state stability, are adapted to the context of nonlinear fractional order differential equations. By studying the input-output stability, researchers can assess the system's resilience to external influences and evaluate its performance in the presence of disturbances or control inputs.

Fractional Lyapunov Exponents:

Fractional Lyapunov exponents provide a powerful tool for characterizing the stability and sensitivity of solutions in nonlinear fractional order differential equations. Fractional Lyapunov exponents measure the average rate of divergence or convergence of nearby trajectories in the system's phase space. These exponents can reveal the presence of chaotic or multi-stable behavior in fractional order systems. By analyzing the spectrum of fractional Lyapunov exponents, researchers can identify regions of stability, instability, and chaotic dynamics. Fractional Lyapunov exponents serve as indicators of system behavior, assisting in the identification of critical parameter values and bifurcation points.

Stability of Fractional Order Systems with Delay:

Many real-world systems exhibit time delays, which introduce additional challenges in stability analysis. Nonlinear fractional order differential equations with delays are particularly relevant in studying systems with memory or feedback loops. Stability analysis of fractional order systems with delay involves considering the combined effect of fractional derivatives and delayed feedback. The stability conditions for such systems are often derived using integral equations or employing delay-dependent stability criteria. Analyzing stability in systems with delay provides insights into the impact of temporal dependencies and time delays on the system's behavior and stability properties.

Numerical Techniques for Stability Analysis:

Stability analysis of nonlinear fractional order differential equations often involves both analytical and numerical techniques. Analytical stability conditions and stability theorems specific to fractional order systems provide theoretical insights into stability properties. However, due to the complexity of fractional order equations and the lack of analytical solutions for many cases, numerical methods play a crucial role in stability analysis. Numerical simulations, such as numerical integration schemes, bifurcation analysis, and time-domain simulations, allow researchers to explore the stability behavior of fractional order systems under various scenarios. Numerical stability analysis techniques enable the examination of stability regions, identification of bifurcations, and characterization of complex dynamics.

In summary, stability analysis is essential in understanding the behavior and robustness of solutions in nonlinear fractional order differential equations. Lyapunov stability analysis, input-output stability analysis, fractional Lyapunov exponents, and stability analysis in systems with delay are key techniques used to assess the stability properties of fractional order systems. Combining analytical stability conditions with numerical simulations enables a comprehensive investigation of stability regions, bifurcations, and complex dynamics. By studying stability, researchers can gain insights into the system's long-term behavior, predict its response to perturbations, and design control strategies to stabilize or manipulate the system's dynamics.

NUMERICAL METHODS AND APPROXIMATION TECHNIQUES:

Numerical methods and approximation techniques play a crucial role in the study of nonlinear fractional order differential equations. Due to the inherent complexity and lack of closed-form solutions for many fractional order equations, numerical approaches provide practical tools for analyzing and understanding the behavior of these systems. This section explores various numerical methods and approximation techniques commonly employed in the context of nonlinear fractional order differential equations.

Numerical Integration Methods:

For the purpose of numerically solving fractional order differential equations, numerical integration techniques are essential building blocks. These techniques get close to solving the problem by discretizing the fractional derivatives and then integrating the difference equations that are produced as a consequence. Among the most common techniques for carrying out numerical integration are the Grunwald-Letnikov method, the Caputo method, the Adams-Bashforth method, and the backward differentiation formula. Researchers are now able to analyse the behaviour of fractional order systems over time as well as calculate approximate solutions thanks to these methodologies. The technique of numerical integration that is used is determined by a number of different criteria, including the particular parameters of the fractional order equation that is being solved as well as its accuracy, stability, and computing efficiency.

Fractional Finite Difference Methods:

Fractional finite difference methods provide a discrete approximation of fractional derivatives using finite difference operators. These methods discretize the domain and approximate fractional derivatives at grid points. Common fractional finite difference schemes include the central difference scheme, the forward difference scheme, and the backward difference scheme. By employing fractional finite difference methods, researchers can convert fractional order differential equations into systems of algebraic equations, which can be solved using standard numerical techniques. Fractional finite difference methods are particularly useful in the context of spatially dependent fractional order differential equations, where they enable the analysis of systems with non-local interactions and memory effects.

Spectral Methods:

Spectral methods involve representing the solution of a fractional order differential equation as a series expansion in terms of orthogonal basis functions. The choice of basis functions, such as Legendre polynomials, Chebyshev polynomials, or Fourier series, depends on the specific problem and the properties of the solution. Spectral methods provide high accuracy and convergence rates, especially for smooth solutions. By truncating the series expansion, numerical solutions can be obtained and analyzed. Spectral methods are advantageous for problems with smooth solutions or for capturing specific features of the system, such as oscillatory behavior or boundary conditions.

Finite Element Methods:

Finite element methods are widely used for solving partial differential equations, including fractional order differential equations. These methods divide the domain into a finite number of subdomains or elements and approximate the solution using piecewise-defined basis functions within each element. By constructing a system of equations based on the variational principle or the Galerkin method, finite element methods provide numerical solutions for fractional order systems. Finite element methods are flexible and versatile, allowing researchers to handle complex geometries and boundary conditions. They are particularly useful for analyzing spatially distributed fractional order systems and investigating the impact of different parameters and boundary conditions on the system's behavior.

Approximation Techniques:

The study of nonlinear fractional order differential equations makes use of a variety of approximation techniques in addition to numerical approaches. Padé approximations, fractional Taylor series expansions, and fractional order orthogonal polynomials are some of the approaches that fall under this category. Techniques of approximation provide analytical tools that may be used to generate approximate solutions or derive simplified models that represent the key dynamics of a system. When closed-form solutions are not accessible or when looking for reduced-order models that maintain essential system characteristics, these approximations are especially beneficial since they provide an alternative.

In conclusion, the study of nonlinear fractional order differential equations requires the use of approximation techniques as well as numerical approaches. Numerous strategies are available for finding approximate solutions and researching the behaviour of fractional order systems. These strategies include numerical integration techniques, fractional finite difference methods, spectral methods, and finite element methods. In addition, approximation approaches provide analytical tools that may be used to derive reduced models or generate answers that are approximately correct. The unique features of the issue at hand, the required accuracy, the computing efficiency, and the type of the fractional order equation that is being analysed all play a role in determining which numerical approach or approximation technique is the most suited.

CONCLUSION:

In summing up, the study of nonlinear fractional order differential equations provides a research topic that is both fruitful and difficult to navigate. Within the scope of this review study, five important facets of these equations were investigated: their mathematical formulation and modelling, qualitative analysis, stability analysis, numerical approaches, and approximation techniques. Researchers build methods for evaluating the behaviour of fractional order systems and obtain a greater knowledge of the numerous phenomena displayed by fractional order systems as a result of their exploration of these areas. The review emphasises how important it is to investigate the existence of solutions and the uniqueness of those solutions, to analyse fixed points and periodic orbits, to comprehend attractors and stability, to investigate bifurcation phenomena, and to employ numerical and approximation techniques in order to analyse these equations. In addition, the review emphasises the restrictions and difficulties involved with the study of nonlinear fractional order differential equations, highlighting the need of having analytic methods that are both reliable and effective. The information that can be obtained through the study of these equations may have practical applications in a variety of scientific, engineering, and transdisciplinary domains as research that is now being conducted continues to expand our understanding of these equations. This review article, in its whole, offers a complete overview of the many phenomena and analytic approaches related with nonlinear fractional order differential equations, therefore laying the groundwork for future developments in this fascinating and fast advancing field of study.

REFERENCES:

[1]. Podlubny, I. (1999). Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications (Vol. 198). Academic Press.

[2]. Diethelm, K. (2010). The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type. Springer Science & Business Media.

[3]. Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). Theory and Applications of Fractional Differential Equations (Vol. 204). Elsevier.

[4]. Baleanu, D., Diethelm, K., Scalas, E., & Trujillo, J. J. (2012). Fractional Calculus: Models and Numerical Methods. World Scientific Publishing Co.

[5]. Agrawal, O. P. (2002). Formulation of Euler-Lagrange equations for fractional variational problems. Journal of Mathematical Analysis and Applications, 272(1), 368-379.

[6]. Meerschaert, M. M., & Sikorskii, A. (2011). Stochastic Models for Fractional Calculus. Walter de Gruyter.

[7]. Hilfer, R. (Ed.). (2000). Applications of Fractional Calculus in Physics (Vol. 13). World Scientific Publishing Co.

[8]. Oldham, K. B., & Spanier, J. (1974). The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order. Academic Press.

[9]. Tarasov, V. E. (2011). Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer Science & Business Media.

[10]. Du, R., Cao, J., & Li, X. (2015). Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications (Vol. 88). Springer Science & Business Media.