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A SUMMARY OF A MATRIX'S PRIMARY DECOMPOSITION AND POLYNOMIALS

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Abstract:

The purpose of this paper is to investigate some unanswered questions regarding the primary decomposition of matrices over a field K and to provide an analogous of some well-known results of spectral, algebraic, and geometric multiplicity order of an eigenvalue to any P-component of the characteristic polynomial CA of a matrix A over a field K. Additionally, the paper will attempt to answer some questions that have not yet been asked regarding the primary decomposition of matrices To be more specific, we calculate the dimension of the kernel of a polynomial of a square matrix A over any arbitrary commutative field K in terms of its invariant fac- tors. This allows us to determine the exact size of the kernel. In this application, we get the value of the P-algebraic and P-geometric multiplicity order of any P-component of the characteristic polynomial CA of a matrix A. This is done so that we may use it.

Keywords: Primary decomposition, invariant factors, algebraic multiplicity, geometric multiplicity.

Introduction:

Let's say that K is a field. Let us assume that A is a multiple of Mn(K) and that P is an irreducible polynomial of K[X]. If the characteristic polynomial CA of A is a power of P, then we shall refer to A as a primary matrix that is P-symmetric. According to the Primary Decomposition Theorem, if A Mn(K) is a non-zero matrix and mA(X) = Qs i=1 P I I is the prime decomposition of its minimal polynomial mA(X), then the matrix A is comparable to a block diagonal of P-primary matrices diag. The Primary Decomposition Theorem states that if A Mn(K) (A1, A2, ..., As). It is currently unclear what the dimension of sequence vector spaces called Ker Ps (A) is. In the first part of this paper, we use some in-depth results on module theory over a PID to compute the dimension of the kernel of a polynomial of a square matrix A over a commutative field K in terms of its invariant factors. This is done by using a square matrix A. In the second part of this paper, we use these results to compute the dimension of a kernel of a polynomial of a In the second part, we give the analogous of some well-known results of spectral, algebraic, and geometric multiplicity order of an eigenvalue, to any Pcomponent of the characteristic polynomial CA of a matrix A over any arbitrary commutative field K. This analogy is based on the fact that these results can be applied to any eigenvalue. The P-algebraic multiplicity order and the P-geometric multiplicity order both see some new conclusions produced here as well.

Preliminary Notes:

Let's say that K is a field. Let us assume that M is a vector space with finite dimensions over K, and that f is an endomorphism of M that is a K. The structure of a K[X]-module may be obtained by the endomorphism f by X.m = f(m) for each m that is less than M. This structure is bestowed onto the vector space M. We shall refer to the K[X]-module on M that is induced by f as Mf from now on. Because the ring K[X] is a PID, the following very helpful conclusion may be inferred by applying the structure theorem of finitely produced torsion modules over a PID (see [[6], 2, p. 556], [[8], 14], [[1], p. 235], and [3]):

Theorem 2.1 (Rational canonical form) *Let M be a finite-dimensional vector space over a field K and f be a K-endomorphism of M. Let M_f be the $K[X]$-module induced by f then there exists a unique sequence of polynomials $q_1, \cdots, q_r$ such that:*

$$M_f \simeq \frac{K[X]}{(q_1)} \oplus \frac{K[X]}{(q_2)} \oplus \cdots \oplus \frac{K[X]}{(q_r)}$$

and

- $q_i \mid q_{i+1}$
- $q_r = m_f(X)$ *the minimal polynomial of* f *and* $\prod_{i=1}^{r} q_i = c_f(X)$ *the characteristic polynomial of* f.

The ascending sequence of polynomials $q_1, \cdots, q_r$ are unique and called the invariant factors of f.

If $q_1, \cdots, q_r$ are the invariant factors of f then we will write $IF(f) = (q_1, \cdots, q_r)$.

Let $A \in \mathcal{M}_n(K)$ be a no zero matrix, and for any linear transformation that has matrix A relative to some basis, we denote M_A the $K[X]$-module induced by A. Then by theorem2.1:

$$M_A \simeq \frac{K[X]}{(q_1)} \oplus \frac{K[X]}{(q_2)} \oplus \cdots \oplus \frac{K[X]}{(q_r)}$$

to the extent that qi | qi+1, qr = mA(X), which is the minimum polynomial of A, and Qr i=1 qi = cA(X), which is the characteristic polynomial of A. Invariant factors of A are referred to as the series of polynomials q1, , and qr. The uniqueness and similarity of A's invariant factors are emphasised here. In point of fact, if q1, , and qr are the invariant factors of A, then A is comparable to a block diagonal matrix with the notation diag(A1, A2,..., Am), where Ai = Comp(qi) is the companion matrix of qi. Let's say that K is a field. Let us assume that A is a multiple of Mn(K) and that P is an irreducible polynomial of K[X]. If the characteristic polynomial CA of matrix A is a power of P, then we shall refer to matrix A as a P-primary matrix.

Main Results:

Theorem 3.1 *Let K be a field. Let $A \in \mathcal{M}_n(K)$ be a non zero matrix and $IF(A) = (q_1, \cdots, q_r)$ its invariant factors. Then*

$$dim_K Ker P(A) = \sum_{i=1}^{r} deg\,(gcd(P, q_i))$$

for any $P \in K[X]$. In particular $dim_K Ker A$ is the number of i such that $q_i(0) = 0$.

To prove this Theorem we need the following lemmas

Lemma 3.2 *Let u be an endomorphism of a finite dimensional vector space E over K. Assume that $E = \oplus_{i=1}^{n} E_i$ such that E_i are $u-$invariant subspaces of E. Then $u = \oplus_{i=1}^{n} u_i$ with $u_i = res_{E_i} u$ the restriction of u to E_i and*

- $u(x) = \sum_{i=1}^{n} u_i(x_i)$ *for all* $x = \sum_{i=1}^{n} x_i$.
- $P(u) = \oplus_{i=1}^{n} P(u_i)$ *for all* $P \in K[X]$
- $Ker P(u) = \oplus_{i=1}^{n} Ker P(u_i)$

Proof. Easy to prove (see [[8], Proposition 1. 3. 2] and [5]). ∎

Lemma 3.3 *Let R be a PID and let a, b be nonzero elements of R. If $d = (a, b) = gcd\{a, b\}$, then*

$$\{\overline{c} \in R/bR \mid a\overline{c} = \overline{0}\} \simeq R/dR.$$

Proof. Indeed let $M_a := \{\overline{c} \in R/bR \mid a\overline{c} = \overline{0}\}$ clearly M_a is a submodule of the R-module R/bR. Let $b' = \frac{b}{d}$. Then

$$\begin{array}{cccc} \phi: & R & \longrightarrow & M_a \\ & x & \longmapsto & \overline{b'x} \end{array}$$

ϕ is an R-homomorphism. Notice that $a\overline{b'x} = \overline{b\frac{a}{d}x} = \overline{0}$. So $\overline{b'x} \in M_a$.

Furthermore if $\overline{ax} = \overline{0}$ then $ax \in bR$ so $x \in b'R$. Hence ϕ is an onto homomorphism. $Ker\phi = \{x \in R \mid b'x \in bR\} = dR$. Hence $M_a \simeq R/dR$. ∎

Lemma 3.4 *Let $A \in \mathcal{M}_n(K)$ and let M_A be the $K[X]$-module induced by A. If $M_A \simeq K[X]/(q)$. Let $P \in K[X]$,then*

$$Ker(P(A)) \simeq Ker\widetilde{P(X)}$$

where $\widetilde{P(X)} : K[X]/(q) \to K[X]/(q), \overline{T} \mapsto P(X).\overline{T}$

Proof. Let φ denotes the $K[X]$-isomorphism between M_A and $K[X]/(q)$ We have $m \in KerP(A)$ if and only if $P(A)(m) = 0$ if and only if$\varphi(P(X).m) = \overline{0}$if and only if $\varphi(P(X).m) = \overline{0}$ if and only if $P(X).\varphi(m) = \overline{0}$ if and only if $\widetilde{P(X)}(\varphi(m)) = 0$ if and only if $\varphi(m) \in Ker\widetilde{P(X)}$, where $\widetilde{P(X)} : K[X]/(q) \to K[X]/(q), \overline{T} \mapsto P(X).\overline{T}$ hence $Ker(P(A)) \simeq Ker\widetilde{P(X)}$. ∎

Lemma 3.5 *Let $A \in \mathcal{M}_n(K)$ and let M_A be the $K[X]$-module induced by A. If $M_A \simeq K[X]/(q)$ then for all $P \in K[X]$*

$$Ker(P(A)) \simeq \begin{cases} (0) & if \quad gcd(P,\, q) = 1 \\ K[X]/(D) & if \quad gcd(P, q) = D \end{cases}$$

Proof. By lemma 3.4 and lemma 3.3 we have $Ker\widetilde{P(X)} \simeq K[X]/(D)$ where $D = gcd(P, q)$. ∎

Now let's give the proof of the theorem 3.1

Proof. Let E be a K-vector space of finite dimension. Let $f \in End_K(E)$ and $\mathcal{B}$ a basis of E such that $mat_{\mathcal{B}}(f) = A$. The space E can be viewed as a K[X]-module ($K[X] \times E \longrightarrow E, (P, x) \longmapsto P.x = P(f)(x)$). Then $E = M_f \simeq \bigoplus_{i=1}^{r} K[X]/(q_i)$ as K[X]-modules, where $q_1, q_2, ..., q_r$ are the invariant factors of A . Hence $E = \oplus_{i=1}^{r} E_i$ where E_i's are f-invariant subspaces and $E_i \simeq K[X]/(q_i)$ as K[X]-modules Hence by lemma 3.2 $f = \oplus_{i=1}^{r} f_i$ and $P(f) = \oplus_{i=1}^{r} P(f_i)$ where $f_i = res_{E_i} f$. So it turns to study the case where f admits one invariant factor (A is companion). By lemma 3.5 $KerP(f_i) \simeq K[X]/(D_i)$ where $gcd(P, q_i) = D_i$. We have by lemma 3.2 $KerP(f) = \oplus_{i=1}^{r} KerP(f_i) \simeq \oplus_{i=1}^{r} K[X]/(D_i)$. Hence $dim_K KerP(f) = \sum_{i=1}^{r} dim_K(K[X]/(D_i)) = \sum_{i=1}^{r} deg\,(D_i) = \sum_{i=1}^{r} deg\,(gcd(P,\, q_i))$. ∎

Corollary 4.7 *Let $f \in End_K(E)$ factors. Let $P \in K[X]$ be an irreducible factor of C_f. Let $s = \upsilon_P(m_f)$. Then $dim_K Ker(f - \lambda I)$ is the number of i such that $q_i(\lambda) = 0$. If further $s = 1$ then the geometric multiplicity order of λ is $\upsilon_P(C_f)$.*

Proof. If $P = X - \lambda$ then by theorem 3.1 we have $dim_K Ker(f - \lambda I) = \sum_{i=1}^{r} deg\,(gcd(X - \lambda, q_i)) = number\,of\,i\,such\,that\,q_i(\lambda) = 0.$
If $s = 1$ we apply the corollary 4.6. ∎

Proposition 4.8 *Let $f \in End_K(E)$. Let $P \in K[X]$ be an irreducible monic factor of C_f. Then $\nu_{alg}(P) = \nu_{geom}(P)$ if and only if $\upsilon_P(m_f) = 1$.*

Proof. Indeed, if $t = \upsilon_P(C_f)$ and $\upsilon_P(m_f) = 1$ then by corollary 4.6 $\nu_{geom}(P) = t degP = \nu_{alg}(P)$. Conversely if $\nu_{alg}(P) = \nu_{geom}(P)$ then $(\sum_{i=1}^{k} s_i + (r - k))degP = t degP$ and hence $\sum_{i=1}^{k} s_i + (r - k) = t$. If $k < r$ then $\sum_{i=k+1}^{r} s_i = r - k$ and $1 < s_i$ for any $k < i$. But the sum $\sum_{i=k+1}^{r} s_i = r - k$ contradicts $1 < s_i$ for any $k < i$. Therefore $k = r$ and $s_r \leq 1$. As P is a component of the characteristic polynomial C_f of f we conclude that $\upsilon_P(m_f) = s_r = 1$. ∎

Proposition 4.9 *Let $f \in End_K(E)$. Let $P \in K[X]$ be an irreducible monic factor of C_f. Then $\nu_{geom}(P) = degP$ if and only if $\upsilon_P(m_f) = \upsilon_P(C_f)$.*

Proof. Indeed $\nu_{geom}(P) = l\,degP$ where $l = \sum_{i=1}^{k} s_i + (r - k)$ and k is the number of indices i such that $s_i \leq 1$. If $\nu_{geom}(P) = degP$ then $l = 1$ hence if $k = r$ then $\sum_{i=1}^{r} s_i = 1$ then $s_r = 1$ and $s_i = 0, \ \forall i \leq r - 1$ since the sequence s_i is non negative and increasing. So $\upsilon_P(m_f) = 1 = \upsilon_P(C_f)$.
If $k < r$ then $l = \sum_{i=1}^{k} s_i + (r - k) = 1$ implies that $k = r - 1$ and $s_i = 0\ \forall i \leq r - 1$. Hence $\upsilon_P(C_f) = \sum_{i=1}^{r} s_i = s_r = \upsilon_P(m_f)$.
Conversely if $\upsilon_P(m_f) = \upsilon_P(C_f)$ then $\sum_{i=1}^{r-1} s_i = 0$ so $s_i = 0\ \forall i \leq r-1$. If $k < r$ then $k = r - 1$ so $\nu_{geom}(P) = (\sum_{i=1}^{r-1} s_i + (r - (r - 1)))degP = degP$. If $k = r$ then $s_r \leq 1$ and since P is a component of the characteristic polynomial C_f of f we conclude that $s_r = 1$ and by consequence $l = 1$ and $\nu_{geom}(P) = degP$.

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