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On Birecurrent Finsler Space for Projective Curvature Tensor

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Abstract:

In this paper, we introduced a Finsler space which W^i_{jkh} satisfies the birecurrence property in sense of Cartan. Further, if the directional derivative of covariant tensor field vanish, then the curvature tensor H^i_{jkh}, associate tensor H_{jskh} and H −Ricci tensor H_{jk} are birecurrent in Affinely connected space.

Keywords: Birecurrence property, Projective curvature tensor, Affinely connected space

Introduction and Preliminaries:

The birecurrent Finsler spaces have been studied by Pandey [8], Dikihi [5], Qasem [10], Qasem and Saleem [12], Muhib [7], Saleem and Abdallah [14-16] and Verma [18]. An affinely connected space for hv −curvature tensor that satisfy the birecurrence property discussed by [6]. Let us consider an n −dimensional Finsler space F_n equipped with the line elements (x, y) and the fundamental metric function F that positive homogeneous of degree one in y^i [1, 3, 13]. The vectors y_i and y^i satisfy

(1.1) a) $y_i y^i = F^2$, b) $\dot\partial_i y_j = \dot\partial_j y_i = g_{ij}$, c) $g_{it} y^i = y_t$ and d) $g_{ij} = \frac{1}{2}\dot\partial_i \dot\partial_j F^2$

Cartan's covariant derivative of the fundamental metric function F, vector y^i and unit vector l^i vanish identically, i.e.

(1.2) a) $F_{|l} = 0$, b) $y^i_{|l} = 0$, and c) $g_{jk|l} = 0$,

Cartan's covariant derivative of an arbitrary tensor T^i_h with respect to x^l is given by [4]

(1.3) a) $\dot\partial_j (T^i_{h|l}) - (\dot\partial_j T^i_h)_{|l} = T^r_h (\dot\partial_j \Gamma^{*i}_{lr}) - T^i_r (\dot\partial_j \Gamma^{*r}_{lj}) - (\dot\partial_r T^i_h) P^r_{jl}$,

where b) $P^r_{jl} = (\dot\partial_j \Gamma^{*r}_{hl}) y^h$ and c) $P^i_{jl} = g^{ih} P_{hjl}$.

The Berwald curvature tensor H^i_{jkh} is positively homogeneous of degree zero in y^i and skew-symmetric in its last two lower indices which defined by [13]

$$H^i_{jkh} = \partial_h G^i_{jk} + G^r_{jk} G^i_{rh} + G^i_{rk} G^r_j - h/k.$$

In view of Euler's theorem on homogeneous functions, we have the following relations

(1.4) a) $\dot\partial_j H^i_{kh} = H^i_{jkh}$, b) $H^i_{jkh} y^j = H^i_{kh}$, c) $H_{ijkh} = g_{jr} H^r_{ikh}$,
d) $H^i_{kh} y^k = H^i_h$, e) $H^i_{kh} = \dot\partial_k H^i_h$, f) $H_{jk} = H^r_{jkr}$,
g) $H_k = H^r_{kr}$, h) $H = \frac{1}{n-1} H^r_r$ and i) $H^r_{rkh} = H_{kh} - H_{hk}$.

The relation between the normal projective curvature tensor N^i_{jkh} and Berwald curvature tenser H^i_{jkh} satisfies [8, 9]

(1.5) $N^i_{jkh} = H^i_{jkh} - \frac{1}{n+1} y^i \dot\partial_j H^r_{rkh}$,

where the normal projective curvature tensor N^i_{jkh} is homogeneous of degree zero in y^i.

Contracting the indices i and j in (1.5) and using the fact that the tensor $H^r_{rkh}$is positively homogeneous of degree zero in y^i, we get

(1.6) $N^r_{rkh} = H^r_{rkh}$.

Transvecting (1.5) by y^j and using (1.4b), we get

(1.7) $N^i_{jkh} y^j = H^i_{kh}$.

The projective curvature tensor W^i_{jkh} and normal projective curvature tensor N^i_{jkh} are connected [13] by

(1.8) a) $W^i_{jkh} = N^i_{jkh} + 2(\delta^i_k M_{hj} - M_{kh}\delta^i_j - k|h)$,

where b) $M_{kh} = -\frac{1}{n^2-1}(nN_{kh} + N_{hk})$ and c) $N_{jk} = N^r_{jkr}$.

The projective curvature tensor W^i_{jkh}satisfies the following [13]

(1.9) a) $W^i_{jkh} y^j = W^i_{kh}$, b) $W^i_{kh} y^k = W^i_h$ and c) $W^i_h y^h = 0$.

A Finsler space whose connection parameter G^i_{jk} is independent of y^i is called an *affinely connected space* [13]. Thus, one of the equivalent equations characterizes an affinely connected space

(1.10) a) $G^i_{jkh} = 0$ and b) $C_{ijk|h} = 0$.

The connection parameters of Cartan and Berwald Γ^{*i}_{kh} and G^i_{jk} coincide in affinely connected space and they are independent of the direction argument, i.e. [2, 11]

(1.11) a) $\dot{\partial}_j G^i_{kh} = 0$ and b) $\dot{\partial}_j \Gamma^{*i}_{kh} = 0$.

Cartan's connection parameter Γ^{*i}_{kh} coincides with Berwald's connection parameter G^i_{kh} for a Landsberg space, which is characterized by [13]

(1.12) $y_r G^r_{jkh} = -2C_{jkh|r} y^r = -2P_{jkh} = 0$.

The $W -$ recurrent Finsler space introduced and defined by [17]

(1.13) $W^i_{jkh|l} = \lambda_l W^i_{jkh}$, $W^i_{jkh} \neq 0$.

where λ_l is non-zero covariant vector field.

Main Results

Definition 2.1. Finsler space F_n which the projective curvature tensor W^i_{jkh} satisfies the following birecurrent property i.e. characterized by

(2.1) $W^i_{jkh|l|m} = a_{lm} W^i_{jkh}$, $W^i_{jkh} \neq 0$.

where a_{lm} is non-zero covariant tensor field. This space will be called a W – *Birecurrent Finsler space*. And denote it briefly by $WBR - F_n$.

Transvecting (2.1) by y^j, using (1.2b) and (1.11a), we get

(2.2) $W^i_{kh|l|m} = a_{lm} W^i_{kh}$.

Transvecting (2.2) by y^k , using (1.2b) and (1.9b), we get

(2.3) $W^i_{h|l|m} = a_{lm} W^i_h$.

Thus, we conclude

Theorem 2.1. *In $WBR - F_n$, the projective torsion tensor W^i_{jk} and projective deviation tensor W^i_h are birecurrent.*

Differentiating (1.8a) covariantly with respect to x^l and x^m in the sense of Cartan, we get

(2.4) $N^i_{jkh|l|m} = W^i_{jkh|l|m} + 2(\delta^i_j M_{kh|l|m} + \delta^i_h M_{jk|l|m})$.

Using (2.1) and (1.8a) in above equation, we get

$$N^i_{jkh|l|m} = a_{lm}[N^i_{jkh} - 2(\delta^i_j M_{kh} + \delta^i_h M_{jk})] + 2(\delta^i_j M_{kh|l|m} + \delta^i_h M_{jk|l|m}).$$

Contracting i and h in above equation and using (1.8c) and the property skew – symmetric for M_{jk}, we get

$$N_{jk|l|m} = a_{lm}[N_{jk} - 2(1-n)M_{jk}] + 2(1-n)M_{jk|l|m}.$$

Using (1.8b) in above equation, we get

$$N_{jk|l|m} = a_{lm}N_{jk} - \frac{2}{n+1} a_{lm}(nN_{jk} + N_{kj}) + \frac{2}{n+1}(nN_{jk|l|m} + N_{kj|l|m}).$$

Using the property skew –symmetric for N_{jk} in above equation, we get

$$N_{jk|l|m} = a_{lm}N_{jk} - 2a_{lm}N_{jk} + 2N_{jk|l|m}.$$

which can be written by

(2.5) $N_{jk|l|m} = a_{lm}N_{jk}$.

Thus, we conclude

Theorem 2.2. *In $WBR - F_n$, if M_{jk} and N_{jk} is property skew –symmetric, then N_{jk} is birecurrent.*

Differentiating (1.8b) covariantly with respect to x^l and x^m in the sense of Cartan, using (2.5), we get

(2.6) $M_{jk|l|m} = -\frac{2}{n^2-1} a_{lm}(nN_{jk} + N_{kj})$.

Using (1.8b) in (2.5), we get

(2.7) $M_{jk|l|m} = a_{lm}M_{jk}$.

Using (2.1), (2.7) and (1.8a) in (2.4), we get

(2.8) $N^i_{jkh|l|m} = a_{lm}N^i_{jkh}$.

Thus, we conclude

Theorem 2.3. *In $WBR - F_n$, the tensor M_{jk} and the normal projective curvature tensor N^i_{jkh} are birecurrent.*

Transvecting (2.8) by y^j, using (1.2b) and (1.7), we get

(2.9) $H^i_{kh|l|m} = a_{lm}H^i_{kh}$.

Transvecting (2.9) by y^k, using (1.2b) and (1.4d), we get

(2.10) $H^i_{h|l|m} = a_{lm}H^i_h$.

Contracting the indies i and h in (2.8) and using (1.4g), we get

(2.11) $H_{k|l|m} = a_{lm}H_k$.

Contracting the indies i and h in (2.9) and using (1.4h), we get

(2.12) $H_{|l|m} = a_{lm}H$.

Thus, we conclude

Theorem 2.4. *In $WBR - F_n$, the torsion tensor H^i_{kh}, deviation tensor H^i_h, curvature vector H_k and scalar curvature H are birecurrent.*

In next result, we obtained the necessary and sufficient condition for some tensors to be birecurrent in $WBR - F_n$. Differentiating (2.9) partially with respect to y^j, we get

$$\dot\partial_j(H^i_{kh|l|m}) = (\dot\partial_j a_{lm})H^i_{kh} + a_{lm}\dot\partial_j H^i_{kh}.$$

Using commutation formula exhibited by (1.3a) for H^i_{kh} in above equation, using (1.4a), we get

(2.13) $H^i_{jkh|l|m} + [H^r_{kh}(\dot{\partial}_j \Gamma^{*i}_{rl}) - H^i_{rh}(\dot{\partial}_j \Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j \Gamma^{*r}_{hl}) - H^i_{rkh} P^r_{jl}]_{|m}$

$+H^s_{kh|l}(\dot{\partial}_j \Gamma^{*i}_{sm}) - H^i_{sh|l}(\dot{\partial}_j \Gamma^{*s}_{km}) - H^i_{ks|l}(\dot{\partial}_j \Gamma^{*s}_{hm}) - H^i_{kh|s}(\dot{\partial}_j \Gamma^{*s}_{lm})$

$-H^i_{skh|l} P^s_{jm} = (\dot{\partial}_j a_{lm}) H^i_{kh} + a_{lm} H^i_{jkh}.$

This shows that

(2.14) $H^i_{jkh|l|m} = a_{lm} H^i_{jkh}$

if and only if

(2.15) $[H^r_{kh}(\dot{\partial}_j \Gamma^{*i}_{rl}) - H^i_{rh}(\dot{\partial}_j \Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j \Gamma^{*r}_{hl}) - H^i_{rkh} P^r_{jl}]_{|m} + H^s_{kh|l}(\dot{\partial}_j \Gamma^{*i}_{sm})$

$-H^i_{sh|l}(\dot{\partial}_j \Gamma^{*s}_{km}) - H^i_{ks|l}(\dot{\partial}_j \Gamma^{*s}_{hm}) - H^i_{kh|s}(\dot{\partial}_j \Gamma^{*s}_{lm}) - H^i_{skh|l} P^s_{jm} - (\dot{\partial}_j a_{lm}) H^i_{kh} = 0.$

Transvecting (2.13) by g_{ti}, using (1.4c), (1.1c) and (1.2c), we get

(2.16) $H_{jtkh|l|m} + g_{ti}[H^r_{kh}(\dot{\partial}_j \Gamma^{*i}_{rl}) - H^i_{rh}(\dot{\partial}_j \Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j \Gamma^{*r}_{hl}) - H^i_{rkh} P^r_{jl}]_{|m}$

$+g_{ti}[H^s_{kh|l}(\dot{\partial}_j \Gamma^{*i}_{sm}) - H^i_{sh|l}(\dot{\partial}_j \Gamma^{*s}_{km}) - H^i_{ks|l}(\dot{\partial}_j \Gamma^{*s}_{hm}) - H^i_{kh|s}(\dot{\partial}_j \Gamma^{*s}_{lm}) - H^i_{skh|l} P^s_{jm}]$

$= g_{ti}(\dot{\partial}_j a_{lm}) H^i_{kh} + a_{lm} H_{jtkh}$.

This shows that

(2.17) $H_{jtkh|m|l} = a_{lm} H_{jtkh}$

if and only if

(2.18) $g_{ti}\{[H^r_{kh}(\dot{\partial}_j \Gamma^{*i}_{rl}) - H^i_{rh}(\dot{\partial}_j \Gamma^{*r}_{kl}) - H^i_{kr}(\dot{\partial}_j \Gamma^{*r}_{hl}) - H^i_{rkh} P^r_{jl}]_{|m} + H^s_{kh|l}(\dot{\partial}_j \Gamma^{*i}_{sm})$

$-H^i_{sh|l}(\dot{\partial}_j \Gamma^{*s}_{km}) - H^i_{ks|l}(\dot{\partial}_j \Gamma^{*s}_{hm}) - H^i_{kh|s}(\dot{\partial}_j \Gamma^{*s}_{lm}) - H^i_{skh|l} P^s_{jm} - (\dot{\partial}_j a_{lm}) H^i_{kh}\} = 0$

Contracting the indices i and h in (2.13), using (1.4f) and (1.4g), we get

(2.19) $H_{jk|m|l} + [H^r_{kt}(\dot{\partial}_j \Gamma^{*t}_{rl}) - H_r(\dot{\partial}_j \Gamma^{*r}_{kl}) - H^t_{kr}(\dot{\partial}_j \Gamma^{*r}_{tl}) - H_{rk} P^r_{jl}]_{|m}$

$+H^s_{kt|l}(\dot{\partial}_j \Gamma^{*t}_{sm}) - H_{r|l}(\dot{\partial}_j \Gamma^{*s}_{km}) - H^t_{ks|l}(\dot{\partial}_j \Gamma^{*s}_{tm}) - H_{k|s}(\dot{\partial}_j \Gamma^{*s}_{lm}) - H_{sk|l} P^s_{jm}$

$= (\dot{\partial}_j a_{lm}) H_k + a_{lm} H_{jk}.$

This shows that

(2.20) $H_{jk|l|m} = a_{lm} H_{jk}$

if and only if

(2.21) $[H^r_{kt}(\dot{\partial}_j \Gamma^{*t}_{rl}) - H_r(\dot{\partial}_j \Gamma^{*r}_{kl}) - H^t_{kr}(\dot{\partial}_j \Gamma^{*r}_{tl}) - H_{rk} P^r_{jl}]_{|m} + H^s_{kt|l}(\dot{\partial}_j \Gamma^{*t}_{sm})$

$-H_{r|l}(\dot{\partial}_j \Gamma^{*s}_{km}) - H^t_{ks|l}(\dot{\partial}_j \Gamma^{*s}_{tm}) - H_{k|s}(\dot{\partial}_j \Gamma^{*s}_{lm}) - H_{sk|l} P^s_{jm} = (\dot{\partial}_j a_{lm}) H_k = 0.$

Thus, we conclude

Theorem 2.5. *In $WBR - F_n$, the Berwald curvature tensor H^i_{jkh}, associate tensor H_{jtkh} and $H -$ Ricci tensor H_{jk} are birecurrent if and only if (2.15), (2.18) and (2.21) hold.*

Remark 3.2. If the $WBR - F_n$ is affinely connected space, then the new space will be *called $WBR -$ affinely connected space.*

Let us consider WBR $-$ affinely connected space. In view of (1.3c), (1.11b), (1.12) and if $\dot{\partial}_j a_{lm} = 0$, then (2.13) becomes

(2.22) $H^i_{jkh|m} = \lambda_m H^i_{jkh}.$

In view of (1.3c), (1.11b), (1.12) and if $\dot{\partial}_j a_{lm} = 0$, then (2.16) becomes

(2.23) $H_{jtkh|m|l} = a_{lm} H_{jtkh}.$

Contracting the indices i and h in (2.13), using (1.3c), (1.11b), (1.12) and if $\dot{\partial}_j a_{lm} = 0$, then (2.19) becomes

(2.24) $H_{jk|m|l} = a_{lm} H_{jk}$

Thus, we conclude

Theorem 2.6. *In WBR – affinely connected space, if the directional derivative of covariant tensor field vanish, then the curvature tensor H^i_{jkh}, associate tensor H_{jskh} and H –Ricci tensor H_{jk} are birecurrent.*

Conclusion:

This paper discussed some tensors that are birecurrent in W–birecurrent Finsler space. The necessary and sufficient condition for some tensors that be birecurrent has been discussed.

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