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MODULATION AND PARSEVAL'S THEOREM FOR WAVELET TRANSFORM AS AN EXTENSION OF FRACTIONAL FOURIER TRANSFORM

Shubham D Shedge[1] & Bharat N Bhosale[2]

[1] *Department of Mathematics, Rajarshi Chhatrapati Shahu College, Kolhapur, Maharashtra, India.*
[2] *Department of Mathematics, Prof. Sambhajirao Kadam College, Deur, Satara, Maharashtra, India.*
Corresponding Author - Shubham D Shedge
Email - Shedge.shubham@gmail.com
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Abstract:

Integral transforms have been wide use to solve various ordinary and partial differential equations or problems in pure and applied mathematics. Wavelet Transform and Fractional Fourier transform has many applications in signal and image processing.

This paper describes the scaling, modulation and Parseval's theorem of Wavelet Transform as an extension of Fractional Fourier transform.

Keywords: *Fractional Fourier Transform, Wavelet Transform, Extended Wavelet Transform.*

Introduction:

Integral Transform was successfully used for almost 2 century years for solving many problems in mathematics[6]. There are many integral transforms have been used for solving differential equations[8]. The fractional Fourier analysis is used for investigations of fractal structures; which in turn are used to analyze different physical phenomena[2]. The ordinary Fourier transform and related techniques have importance in many areas of science, engineering and technology[10]. The Fourier transform is best mathematical tool used in differential equations, physical optics, signal and image processing and so on[1,4].

The concept of wavelet started to appeared in the literature only in the 19^{th} century 8^{th} decade that used by Morlet(1982)[3,9]. A French geophysical engineering first introduced the idea of wavelet transform as the mathematical tool for signal and image processing[5]. The wavelet transform decomposes a signal into the representation that shows signal details and tends as a function of time[8]. The kernel of fractional Fourier transform and wavelet transform are closely related to each other so Sharma and bhosale introduce the Wavelet transform as an extension of fractional Fourier transform[7]. so we are going to discuss modulation and Parseval's theorem for Wavelet transform as an extension of fractional Fourier transform.

Preliminaries:

Wavelet Transform as an extension of Fractional Fourier Transform[7]:

The Wavelet transform as the extension of fractional Fourier transform of $f(x) \in E(R^n)$ is denoted by $W\big(f(x)\big)(\xi)$ and defined by,

$$W\big(f(x)\big)(a,b) = W\big(f(x)\big)(\xi)$$

$$= \int_{-\infty}^{\infty} BC_{1\alpha} f(x) e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, dx$$

Where, $b = \xi sec\alpha$, $a = \tan^{\frac{1}{2}}\alpha$, $B = \frac{e^{\frac{i}{2}b^2}\sin^2\alpha}{C(\alpha)|a|^n}$, $C(\alpha) = \frac{e^{\frac{i\alpha}{2}}}{(2\pi i sin\alpha)^{\frac{1}{2}}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}$,

$C_{1\alpha} = (2\pi i sin\alpha)^{-\frac{1}{2}} \exp\left(\frac{i\alpha}{2}\right)$, $0 \le \alpha < \frac{\pi}{2}$.

Testing Function Space $E(R^n)$:

An infinitely differentiable complex valued function f on R^n belongs to $E(R^n)$ if for each compact set $X \subset S_\beta$ where

$S_\beta = \{y \in R^n : |y| \le \beta, \beta > 0\}$

Extended Wavelet Transform of Translation [8]:

$$W\big(f(x - x_0)\big)(\xi)$$

$$= e^{iC_{2\alpha}[x_0^2 cos\alpha - 2x_0\xi]} W\left(e^{(2iC_{2\alpha}x_0 cos\alpha)x} f(x)\right)(\xi)$$

Differentiation of Extended Wavelet Transform[8]:

$$D^n W\big(f(x)\big)(\xi) = \frac{d^n}{d\xi^n} W\big(f(x)\big)(\xi)$$

$$= W\left(\sum_{h=0}^{\left[\frac{n}{2}\right]} C_h C_{\alpha,h} (\xi cos\alpha - x)^{n-2h} f(x)\right)(\xi)$$

Where,

$C_h = \frac{n!}{(n-2h)!h!}(i)^{n-h}(2)^{n-2h}$, $C_{\alpha,h} = (C_{2\alpha})^{n-h}\cos^h\alpha$

Scaling Property of Wavelet Transform as an Extension of Fractional Fourier Transform:

$$W\big(f(ax)\big)(\xi)$$

$$= \frac{1}{a} W\left(e^{-iC_{2\alpha}\left[\left(\frac{a^2-1}{a^2}\right)x^2 - 2\left(\frac{a-1}{a}\right)x\xi\right]} f(x)\right)(\xi)$$

Proof: We Know that,

$$W\big(f(x)\big)(\xi)$$

$$= \int_{-\infty}^{\infty} BC_{1\alpha} f(x) e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, dx$$

$$W\big(f(ax)\big)(\xi) = \int_{-\infty}^{\infty} BC_{1\alpha} f(ax) e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, dx$$

$$= \int_{-\infty}^{\infty} BC_{1\alpha} f(x') e^{iC_{2\alpha}\left[\left(\left(\frac{x'}{a}\right)^2+\xi^2\right)cos\alpha - 2\left(\frac{x'}{a}\right)\xi\right]} \frac{dx'}{a}$$

$=\frac{1}{a}\int_{-\infty}^{\infty} BC_{1\alpha} f(x')e^{iC_{2\alpha}\left[\left(\left(1-\frac{a^2-1}{a^2}\right)x'^2+\xi^2\right)cos\alpha-2\left(1-\frac{a-1}{a}\right)x'\xi\right]}dx'$

$=\frac{1}{a}\int_{-\infty}^{\infty} BC_{1\alpha} f(x)e^{iC_{2\alpha}\left[\left(\left(1-\frac{a^2-1}{a^2}\right)x^2+\xi^2\right)cos\alpha-2\left(1-\frac{a-1}{a}\right)x\xi\right]}dx$

$=\frac{1}{a}\int_{-\infty}^{\infty} BC_{1\alpha} f(x)e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]}\, e^{iC_{2\alpha}\left[\left(\frac{a^2-1}{a^2}\right)x^2-2\left(\frac{a-1}{a}\right)x\xi\right]}dx$

$W(f(ax))(\xi)$

$$=\frac{1}{a}W\left(e^{-iC_{2\alpha}\left[\left(\frac{a^2-1}{a^2}\right)x^2-2\left(\frac{a-1}{a}\right)x\xi\right]}f(x)\right)(\xi)$$

Modulation of Wavelet Transform as an Extension of Fractional Fourier Transform:

I.

$$W(f(x)cosax)(\xi) = \frac{1}{2}\left\{e^{iC_{2\alpha}\left[\left(\frac{a\xi}{C_{2\alpha}}-\frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]}W(f(x))\left(\xi-\frac{a}{2C_{2\alpha}}\right)+e^{iC_{2\alpha}\left[\left(-\frac{a\xi}{C_{2\alpha}}-\frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]}W(f(x))\left(\xi+\frac{a}{2C_{2\alpha}}\right)\right\}$$

Proof:

$W(f(x)cosax)(\xi) = \int_{-\infty}^{\infty} BC_{1\alpha}e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]}f(x)cosax\, dx$

$=\int_{-\infty}^{\infty} BC_{1\alpha}e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]}f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)dx$

$=\frac{1}{2}\{\int_{-\infty}^{\infty} BC_{1\alpha}e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]}f(x)e^{iax}dx + \int_{-\infty}^{\infty} BC_{1\alpha}e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha-2x\xi]}f(x)e^{-iax}dx\}$

$$=\frac{1}{2}\{\int_{-\infty}^{\infty} BC_{1\alpha}e^{iC_{2\alpha}\left[(x^2+\xi^2)cos\alpha-2x\left(\xi-\frac{a}{2C_{2\alpha}}\right)\right]}f(x)dx + \int_{-\infty}^{\infty} BC_{1\alpha}e^{iC_{2\alpha}\left[(x^2+\xi^2)cos\alpha-2x\left(\xi+\frac{a}{2C_{2\alpha}}\right)\right]}f(x)dx\}$$

$$=\frac{1}{2}\{\int_{-\infty}^{\infty} BC_{1\alpha}e^{iC_{2\alpha}\left[\left(x^2+\left(\xi-\frac{a}{2C_{2\alpha}}\right)^2\right)cos\alpha-2x\left(\xi-\frac{a}{2C_{2\alpha}}\right)\right]}f(x)e^{iC_{2\alpha}\left[\left(\frac{a\xi}{C_{2\alpha}}-\frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]}dx + \int_{-\infty}^{\infty} BC_{1\alpha}e^{iC_{2\alpha}\left[\left(x^2+\left(\xi+\frac{a}{2C_{2\alpha}}\right)^2\right)cos\alpha-2x\left(\xi+\frac{a}{2C_{2\alpha}}\right)\right]}f(x)e^{iC_{2\alpha}\left[\left(-\frac{a\xi}{C_{2\alpha}}-\frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]}dx\}$$

$$=\frac{1}{2}\left\{e^{iC_{2\alpha}\left[\left(\frac{a\xi}{C_{2\alpha}}-\frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]}W(f(x))\left(\xi-\frac{a}{2C_{2\alpha}}\right)+e^{iC_{2\alpha}\left[\left(-\frac{a\xi}{C_{2\alpha}}-\frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]}W(f(x))\left(\xi+\frac{a}{2C_{2\alpha}}\right)\right\}$$

II.

$$W(f(x)sinax)(\xi) = \frac{1}{2i}\left\{e^{iC_{2\alpha}\left[\left(\frac{a\xi}{C_{2\alpha}}-\frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]}W(f(x))\left(\xi-\frac{a}{2C_{2\alpha}}\right)-e^{iC_{2\alpha}\left[\left(-\frac{a\xi}{C_{2\alpha}}-\frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]}W(f(x))\left(\xi+\frac{a}{2C_{2\alpha}}\right)\right\}$$

Proof:

$W(f(x)sinax)(\xi) =$

$\int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]} f(x) sinax\, dx$

$= \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]} f(x) \left(\frac{e^{iax} - e^{-iax}}{2i}\right) dx$

$= \frac{1}{2i}\{\int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]} f(x) e^{iax} dx - \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]} f(x) e^{-iax} dx \}$

$$= \frac{1}{2i}\{ \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}\left[(x^2+\xi^2)cos\alpha - 2x\left(\xi - \frac{a}{2C_{2\alpha}}\right)\right]} f(x) dx - \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}\left[(x^2+\xi^2)cos\alpha - 2x\left(\xi + \frac{a}{2C_{2\alpha}}\right)\right]} f(x) dx \}$$

$$= \frac{1}{2i}\{ \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}\left[\left(x^2+\left(\xi-\frac{a}{2C_{2\alpha}}\right)^2\right)cos\alpha - 2x\left(\xi - \frac{a}{2C_{2\alpha}}\right)\right]} f(x) e^{iC_{2\alpha}\left[\left(\frac{a\xi}{C_{2\alpha}} - \frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]} dx - \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}\left[\left(x^2+\left(\xi+\frac{a}{2C_{2\alpha}}\right)^2\right)cos\alpha - 2x\left(\xi + \frac{a}{2C_{2\alpha}}\right)\right]} f(x) e^{iC_{2\alpha}\left[\left(-\frac{a\xi}{C_{2\alpha}} - \frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]} dx \}$$

$$= \frac{1}{2i}\left\{ e^{iC_{2\alpha}\left[\left(\frac{a\xi}{C_{2\alpha}} - \frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]} W(f(x))\left(\xi - \frac{a}{2C_{2\alpha}}\right) - e^{iC_{2\alpha}\left[\left(-\frac{a\xi}{C_{2\alpha}} - \frac{a^2}{C_{2\alpha}^2}\right)cos\alpha\right]} W(f(x))\left(\xi + \frac{a}{2C_{2\alpha}}\right)\right\}$$

Parseval's Theorem for Wavelet Transform as an Extension of Fractional Fourier Transform:

i)

$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} W(f(x))(\xi)\, \overline{W(g(x))(\xi)}\, d\xi$

ii)

$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |W(f(x))(\xi)|^2\, d\xi$

Proof:

i) Let,

$W(g(x))(\xi) =$

$\int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, g(x) dx$

By using inverse formula,

$g(x)$

$$= \int_{-\infty}^{\infty} \overline{BC_{1\alpha}} e^{-iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, W(g(x))(\xi) d\xi$$

$\overline{g(x)}$

$$= \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, \overline{W(g(x))(\xi)} d\xi$$

Now consider,

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\, \overline{W(g(x))(\xi)} d\xi\ dx$$

$= \int_{-\infty}^{\infty} \overline{W(g(x))(\xi)} (\int_{-\infty}^{\infty} BC_{1\alpha} e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]} f(x)\ dx) d\xi$

$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \int_{-\infty}^{\infty} W(f(x))(\xi)\, \overline{W(g(x))(\xi)}\, d\xi$

ii) $\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} f(x)\overline{f(x)}\, dx$

$= \int_{-\infty}^{\infty} W(f(x))(\xi)\, \overline{W(f(x))(\xi)}\, d\xi$

$= \int_{-\infty}^{\infty} |W(f(x))(\xi)|^2\, d\xi$

Conclusion:

This paper presents scaling, modulation and Parseval's theorem for Wavelet transform as an extension of fractional Fourier transform and this are useful to solve ordinary differential equations and partial differential equations like heat equation, schrodinger's equation etc.

References:

1. B.N.Bhosale, Integral transformation of Generalized functions. Discovery publishing hour, 2005.
2. Bhosale, B. (2011). On Fractional Fourier Analysis in Ultra-distributional set up and Image Processing.
3. Bhosale, B., & Biswas, A. (2012). Wavelet analysis of optical solitons and its energy aspects. *Mathematics in Engineering, Science & Aerospace (MESA)*, *3*(1).
4. Sharma, V. D. (2013). Modulation and Parsvels Theorem For Generalized Fractional Fourier Transform. *Engineering Research and Applications*, *3*, 2248-9622.
5. Sharma, V. D., & Rangari, A. N. (2014). Operational Calculus on Generalized Fourier-Laplace Transform. *Int. Journal of Scientific and Innovative Mathematical Research (IJSIMR)*, *2*(11), 862-867.
6. Sharma, V., & Deshmukh, P. (2015). Introduction to two dimensional fractional fourier-mellin transform. *Int J Res Eng Appl Sci*, *5*(12), 137-141.
7. Sharmaa, V., & Bhosaleb, B. On Wavelet Transform: An extension of Fractional Fourier Transform and its applications in optical signal processing.
8. Shedge, S. D., & Bhosale, B. N. OPERATIONAL CALCULUS ON WAVELET TRANSFORM AS AN EXTENSION OF FRACTIONAL FOURIER TRANSFORM.
9. Sifuzzaman, M., Islam, M. R., & Ali, M. Z. (2009). Application of wavelet transform and its advantages compared to Fourier transform.
10. Soni, A., Jain, J., & Roshan, R. (2013, March). Image steganography using discrete fractional Fourier transform. In *2013 International Conference on Intelligent Systems and Signal Processing (ISSP)* (pp. 97-100). IEEE.