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ON JOIN GRAPH OF ZERO-DIVISOR GRAPHS OF DIRECT PRODUCT OF FINITE FIELDS

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Abstract:
I. Beck introduced the concept of Zero-divisor graph of a commutative ring R with all the elements of ring R as vertices and two distinct vertices x, y are adjacent if and only if $x \cdot y = 0$. Anderson and Livingston modified the definition of Zero-divisor graph given by Beck, by considering only the non-zero zero-divisors as the vertices of the Zero-divisor graph denoted by $\Gamma(R)$ and two distinct vertices x, y are adjacent if and only if $x \cdot y = 0$. The Join graph $G + H$ of two graphs G and H is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G),\ v \in V(H)\}$. In this paper we determine the graph properties such as diameter, girth, clique number, vertex chromatic number, independence number of Join graph of Zero-divisor graphs of direct product of finite fields.

Keywords: *Zero-divisor graphs, Join Graph, Clique, and Chromatic number.*
2020 Mathematics Subject Classification: *13A70, 05C15, 05C25, 05C69.*

Introduction:

The concept of Zero Divisor graphs of a commutative ring R, was introduced by I. Beck in [2]. Beck considered all the elements of the ring R as the vertices of the Zero divisor graph and two distinct vertices x and y are adjacent if and only if $x \cdot y = 0$.

This definition of Zero divisor graph given by Beck was modified by Anderson and Livingston [1] in which they considered the non-zero zero divisors of R to be the vertex set of Zero divisor graph of R denoted by $\Gamma(R)$ and two distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $x \cdot y = 0$.

The Join graph G+H of two graphs G and H is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. [4]

In this paper we consider the Zero divisor graph of the Semi-local

Ring, $R = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$, of finite cartesian product of finite fields. Let $Z^*(R)$ be the set of non-zero zero-divisors of the Semi-local ring $R = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$ and $\Gamma(R)$ denote the graph with vertex set as $Z^*(R)$ and edge set as $\{rs : r \cdot s = 0, r, s \in Z^*(R)\}$. For basic graph theoretical terminologies we adopt the definitions of [3], [6].

The distance dG (x, y) of two vertices x, y in a graph G is the length of a shortest $x - y$ path in G. If no such path exists, we set $dG(x, y) = \infty$. The greatest dis- tance between any two vertices in a graph G is called the diameter of G, denoted by diam(G). The minimum length of a cycle contained in a graph G, is called the girth of G. If the graph does not contain any cycles, its girth is defined to be infinity. A subset $S \subset V(G)$ is said to be a Clique in G if every two distinct vertices in S are adjacent in G. A clique S of a graph G is said to be a maximum clique, if there is no clique in G with more vertices. The clique number of a graph G, denoted $\omega(G)$, is the size of the maximal clique of G. A set S of vertices in a graph G, in which no two vertices are adjacent is said to be an independent set and the size of largest possible size of an independent set in a graph is called its independence number and is denoted by $\alpha(G)$. A $k-$vertex coloring of a graph G is an assignment of k colors to the vertices of G such that no two adjacent vertices receive the same color. The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colors required to color the vertices of G such that no two adjacent vertices receive the same color. In this paper we prove that the Clique number and Vertex chromatic number are equal for the Join graph of the zero divisor graphs of finite direct product of finite fields.

Girth and Diameter:

We consider the semi-local rings $R_1 = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$ and $R_2 = J_1 \times J_2 \times \cdots \times J_m$, $(m \geqslant 2)$ where F_i, $(1 \leqslant i \leqslant n)$, J_k, $(1 \leqslant k \leqslant m)$ are finite fields with $|F_i| \geqslant 2$, $|J_k| \geqslant 2$.

In Theorem 2.1, we determine the Diameter and Girth of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$.

Theorem 2.1. Let $R_1 = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$ and $R_2 = J_1 \times J_2 \times \cdots \times J_m$, $(m \geqslant 2)$ where F_i, $(1 \leqslant i \leqslant n)$, J_k, $(1 \leqslant k \leqslant m)$ are finite fields with $|F_i| \geqslant 2$, $|J_k| \geqslant 2$. Then,

(i) the diameter of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 1 if $n = 2$, $m = 2$, $|F_1| =$

$|F_2| = |J_1| = |J_2| = 2$

(ii) the diameter of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 2 if $n = 2$, $m = 2$, $|F_1| \geqslant 3$ or $|F_2| \geqslant 3$ and $|J_1| \geqslant 3$ or $|J_2| \geqslant 3$.

(iii) the diameter of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 3 if $n \geqslant 3$, $m \geqslant 3$. (iv) the girth of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 3 if $n \geqslant 2$ or $m \geqslant 2$.

Proof. (i) Let $n = 2$, $m = 2$, $|F_1| = |F_2| = |J_1| = |J_2| = 2$. Then $\Gamma(R_1)$ and $\Gamma(R_2)$ are both complete graph K_2.[1] Therefore, the join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is complete graph K_4. Hence, the diameter of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 1.

(ii) Let $n = 2$, $m = 2$, $|F_1| \geqslant 3$ or $|F_2| \geqslant 3$ and $|J_1| \geqslant 3$ or $|J_2| \geqslant 3$. Then $\Gamma(R_1)$ is complete bipartite graph $K_{|F_1|-1,|F_2|-1}$ and $\Gamma(R_2)$ is complete bipartite graph $K_{|J_1|-1,|J_2|-1}$.[1] Therefore, diameter of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 2. Consider, the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$. If $x, y \in \Gamma(R_1)$, then $d(x, y) \leqslant 2$. If $x, y \in \Gamma(R_2)$, then $d(x, y) \leqslant 2$. If $x \in \Gamma(R_1)$, and $y \in \Gamma(R_2)$, then $d(x, y) = 1$ by using definition of Join graph. Therefore, the maximum distance between any two distinct vertices in the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 2. Hence, $diam(\Gamma(R_1) + \Gamma(R_2)) = 2$, if $n = 2$, $m = 2$, $|F_1| \geqslant 3$ or $|F_2| \geqslant 3$ and $|J_1| \geqslant 3$ or $|J_2| \geqslant 3$.

(iii) Let $n \geqslant 3$, $m \geqslant 3$. Let x contains $'0'$ in the i^{th} co-ordinate position and $'1'$ in the remaining positions and y contains $'0'$ in the j^{th} co-ordinate position and $'1'$ in the remaining positions, $(1 \leqslant i = j \leqslant n)$. x is not adjacent to y in $\Gamma(R_1)$. x is adjacent to a vertex u containing $'1'$ in the i^{th} co-ordinate position and $'0'$ in the remaining positions and y is adjacent to a vertex v containing $'1'$ in the j^{th} co- ordinate position and $'0'$ in the remaining positions. Also, $i = j \implies$ u is adjacent to v. Therefore, $x - u - v - y$ is a path of length 3. Therefore, $d(x, y) = 3$ if $n \geqslant 3$, and therefore by [1](Theorem 2.3), diam $(\Gamma(R_1)) = 3$. Similarly, diam $(\Gamma(R_2)) = 3$. Consider, the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$. If $x, y \in \Gamma(R_1)$, then $d(x, y) \leqslant 3$. If $x, y \in \Gamma(R_2)$, then $d(x, y) \leqslant 3$. If $x \in \Gamma(R_1)$ and $y \in \Gamma(R_2)$ then $d(x, y) = 1$ by definition of Join graph. Therefore, the maximum distance between any two distinct vertices in the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 3. Hence,

$\operatorname{diam}(\Gamma(R_1)+\Gamma(R_2)) = 3$, if $n \geqslant 3, m \geqslant 3$.

(iv) Let $x, y \in \Gamma(R_1)$ such that $d(x, y) = 1$. If $z \in \Gamma(R_2)$, then x, y are both adjacent to $z \in \Gamma(R_2)$. Thus, $x - y - z - x$ is a cycle of length 3. Therefore, girth of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is 3.

Clique number, Vertex Chromatic number, Independence number:

In Theorem 3.1, we determine the clique number and vertex chromatic number of

Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$.

Theorem 3.1. Let $R_1 = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$ and $R_2 = J_1 \times J_2 \times \cdots \times J_m$, $(m \geqslant 2)$ where F_i, $(1 \leqslant i \leqslant n)$, J_k, $(1 \leqslant k \leqslant m)$ are finite fields. Then,

(i) the clique number of Join Graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $\omega(\Gamma(R_1)+\Gamma(R_2)) = n+m$. (ii) the vertex chromatic number of Join Graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $\chi(\Gamma(R_1) + \Gamma(R_2)) = n + m$.

Proof. (i) By [5] Theorem 2.3, the clique number $\omega(\Gamma(R_1)) = n$, $\omega(\Gamma(R_2)) = m$. If S_1 is the maximal clique set in $\Gamma(R_1)$ with $|S_1| = n$ and S_2 is the maximal clique set in $\Gamma(R_2)$ with $|S_2| = m$, then $S_1 \cup S_2$ is the clique set of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ with $|S_1 \cup S_2| = n + m$. We prove that $S_1 \cup S_2$ is a maximal clique in the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$. If a vertex $x \in (\Gamma(R_1) + \Gamma(R_2)) \setminus (S_1 \cup S_2)$ is adjacent to each and every vertex in $S_1 \cup S_2$, then the clique number of $\Gamma(R_1)$ is at least $n + 1$ and the clique number of $\Gamma(R_2)$ is at least $m + 1$. This is a contradiction to $\omega(\Gamma(R_1)) = n$, $\omega(\Gamma(R_2)) = m$. Therefore, $S_1 \cup S_2$ is a maximal clique in the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$. Therefore, the clique number of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $n + m$.

(ii) Now we prove that the vertex chromatic number of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $n+m$. Since the vertex chromatic number is greater than or equal to clique number, therefore, $\chi(\Gamma(R_1) + \Gamma(R_2)) \geqslant \omega(\Gamma(R_1) + \Gamma(R_2)) = n + m$. By [5] Theorem 2.3 the vertex chromatic number $\chi(\Gamma(R_1)) = n$, $\chi(\Gamma(R_2)) = m$. Therefore, the vertex set $V(\Gamma(R_1))$ can be partitioned into n disjoint independent sets $V_i (1 \leqslant i \leqslant n)$ and vertex set $V(\Gamma(R_2))$ can be partitioned into m disjoint independent sets U_j $(1 \leqslant j \leqslant m)$. It can be easily observed that $V_1 \cup \cdots \cup V_n \cup U_1 \cup \cdots \cup U_m$ is the partition of the vertex set of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ and moreover each set $V_i (1 \leqslant i \leqslant n)$ and each set U_j $(1 \leqslant j \leqslant m)$ are independent sets in the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$. Therefore, the vertex chromatic number $\chi(\Gamma(R_1) + \Gamma(R_2)) \leqslant n + m$. Hence, the vertex chromatic number $\chi(\Gamma(R_1) + \Gamma(R_2)) = n + m$.

Remark 3.2. The clique number and chromatic number are equal for the family of zero divisor graphs of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ of the semi-local rings

$R_1 = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$ and

$R_2 = J_1 \times J_2 \times \cdots \times J_m$, $(m \geqslant 2)$.

Now we determine the independence number of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$.

Theorem 3.3. Let $R_1 = F_1 \times F_2 \times \cdots \times F_n$, $(n \geqslant 2)$ and $R_2 = J_1 \times J_2 \times \cdots \times J_m$, $(m \geqslant 2)$ where F_i, $(1 \leqslant i \leqslant n)$, J_k, $(1 \leqslant k \leqslant m)$ are finite fields. Then, the independence number of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $\alpha(\Gamma(R_1) + \Gamma(R_2)) = \max\{\alpha(\Gamma(R_1)), \alpha(\Gamma(R_2))\}$, where $\alpha(\Gamma(R_1))$ is independence number of $\Gamma(R_1)$ and $\alpha(\Gamma(R_2))$ is independence number of $\Gamma(R_2)$.

Proof. Let S_1 be the independent set in $\Gamma(R_1)$ of size $\alpha(\Gamma(R_1))$ and S_2 be the independent set in $\Gamma(R_2)$ of size $\alpha(\Gamma(R_2))$. Since each vertex in independent set S_1 is adjacent to each and every vertex in independent set S_2 in the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$, therefore $S_1 \cup S_2$ cannot be an independent set in the Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$. Therefore, the independence number of Join graph of $\Gamma(R_1)$ and $\Gamma(R_2)$ is $\alpha(\Gamma(R_1) + \Gamma(R_2)) = \max\{\alpha(\Gamma(R_1)), \alpha(\Gamma(R_2))\}$.

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