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Hybrid Expansion Law for Relativistic Hydrodynamic in $f(R,G)$ Gravity Cosmological Model

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Abstract:

Present investigation devoted towards the analysis of isotropic homogeneous FRW Cosmological model with modified form of $f(R,G)$ gravity (where $f(R,G)$ is a generic function of the Ricci scalar R and Gauss–Bonnet invariant G) in the presence of relativistic hydrodynamic fluid. The exact solution of relativistic hydrodynamic model is obtained by considering the model $f(R,G)=R^{\alpha}G^{\beta}$ gravity and using hybrid expansion law also, analyzed the obtained results graphically. Moreover, some physical and geometrical parameters are also discussed in details.

Keyword: Relativistic Hydrodynamic; $f(R,G)$ gravity; FRW model; Cosmology.

Introduction:

The results coming from the analysis of type Ia supernovae (SNeIa) surveys, large scale structure (LSS) and cosmic microwave background (CMB) anisotropy spectrum strongly indicate that our universe is spatially flat and has a phase transition from decelerating to accelerating [1-5]. It is well known that General Theory of Relativity based on the Einstein-Hilbert action and it cannot explain the acceleration of the early and late universe. Therefore, General Theory of Relativity does not describe precisely gravity and it is quite reasonable to modify it, in order to get this phenomenon, there are several alternatives. First alternative obtained by substituting Einstein-Hilbert term with an arbitrary function of the curvature scalar R, this is the so-called $f(R)$ gravity. This theory has been widely studied and interesting results have been found by [6-8]. Later on, a maximal extension of the Hilbert-Einstein action by assuming that the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and the matter Lagrangian L_m so called $f(R,T)$ gravity where T represents the trace of the stress-energy tensor. This $f(R,T)$ theory of gravity explains the late time cosmic accelerated expansion of the universe and can be used to explore many issues and may provide some satisfactory results. Within the frame work of $f(R,T)$ gravity numbers of authors presented in [9-18] have investigated several aspects of this theory. Chandel and Ram [19] generated new classes of solutions of field equations starting from known solutions for an anisotropic Bianchi type-III cosmological model with perfect fluid in $f(R,T)$ gravity whereas Chaubey et al [20] has obtained a new class of Bianchi type cosmological models in $f(R,T)$ gravity. Among the various modifications of Einstein's theory, another one way to look at the theory beyond the Einstein equation is the Teleparallel Gravity (TG) which uses the Weitzenbock connection in place of the Levi–Civita connection and so it has no curvature but has torsion which is responsible for the acceleration of the Universe. Some relevant works in this gravity are the graphical representation of k-essence with the help of Equation of State parameter describe by Sharif & Rani [21]. In $f(T)$ gravity the existence of relativistic stars investigated by Bohmer *et al.* [22]. Gamal & Nashed [24] investigated anisotropic models with two fluids in linear and quadratic forms of $f(T)$ gravitational models. In the same way the models represent its spreading as accelerated with inflationary era in the early and the very late time of the Universe. For suitably large time resulting model expects that the anisotropy of the Universe will damp out and the Universe will become isotropic. Very recently, Pawar *et al.* [27] the author who have studied the modified holographic Ricci DE model in $f(R,T)$ theory of gravity by using anisotropy Bianchi type Universe and observed that the model has no physical singularity, the Universe is expanding and accelerating exponentially. $f(R)$ and $f(G)$ gravity have generalizations offered by higher-order gravities which use combinations of higher-order curvature invariants constructed from the Ricci and Riemann tensors. Also, the theory which combines Ricci scalar and Gauss-Bonnet scalar called $f(R,G)$ gravity theory. Alvaro and Diego [28] focused on the analysis of $f(R,G)$ gravity and a deep analysis has been performed on the stability of some important cosmological solutions which not only convince to constrain the form of the gravitational action, but also further help in better understanding of the perturbation's behavior in the higher-order theories of gravity.

As, relativistic hydrodynamics is one of the most important part of physics which deal with the study of both, flows in which the velocities attained by the particle or by fluid which is comparable with the speed of light or the intensity of gravitational field strength-either background gravitational field or that generated by the matter itself, or both. The study of the dynamic and evolution of such relativistic astrophysical systems has confidence on the assistance of accurate, large-scale numerical simulations. A relativistic investigation of the matter hydrodynamics and gravity is also important in present scenarios involving the gravitational collapse of massive stars to form black holes, or at the last stage of the coalescence of compact neutron stars binaries. The wealth of latest information that could be extracted thereof and it is one of the driving motivations of research in relativistic astrophysics. The aim of this paper is to present a brief overview of the spatially homogeneous and isotropic FRW cosmological model in $f(R,G)$ gravity with general relativistic hydrodynamics. Alvaro and Diego [29] achieved the ground study on the stability of some significant cosmological solutions which is not only influence to constrain the form of the gravitational action, but it is also help in better accepting of the perturbations comportment in modified $f(R,G)$ theory which will lead to a more particular exploration of the full spectrum of cosmological perturbations. Makarenko et al [30] accepted that the phantom type cosmological model which do not lead a future singularity and defined the cosmological reconstruction in modified $f(R,G)$ Gauss-Bonnet gravity. Atazadeh and Darabi [31] explained the viability of $f(R,G)$ gravity by considering two forms of it by imposing the energy conditions, which account for the stability of the model solutions and they also construct the inequalities derived by energy conditions.

Thereafter applying the weak energy condition taking the recent estimated values of the Hubble, deceleration, jerk and snap parameters to probe the viability. The weak filed of modified $f(R,G)$ gravity taking into consideration analytic function of the R (Ricci scalar) and G (Gauss-Bonnet invariant), precisely they developed in metric formalism. The Newtonian, Post Newtonian and Parameterized Post Newtonian limits starting from general $f(R,G)$ Lagrangian discussed in detail by Laurentis and Lopez-Revelles [32] considering special cases and achieved the Newtonian limit of $f(R,G)$. Lorenzo Sebastiani [33] studied finite time future singularities in framework of $f(R,G)$ gravity and he also discussed a possible way to cure the finite time future singularities by taking into account higher order curvature corrections or effects of viscous fluids.

$f(R,G)$ Gravity with Relativistic Hydrodynamic Source:

The most general action for $f(R,G)$ gravity is given as [34]

$$S=\frac{1}{2k}\int d^4x\sqrt{-g}\big(R+f(G)\big)+S_M(g^{mn},\varphi), \tag{1}$$

where $S_M(g^{mn},\varphi)$ is the matter action, R is Ricci scalar and G is Gauss-Bonnet invariant defined by

$$G=R^2-4R_{mn}R^{mn}+R_{mnrs}R^{mnrs}, \tag{2}$$

where the notations R_{mn} and R_{mnrs} are occupied for the Ricci and Riemann tensors respectively.

Variation of the standard action (1) with respect to the metric gives us the following gravitational field equation:

$$R_{mn}-\frac{1}{2}g_{mn}R=kT^{pat}_{mn}+\Sigma_{mn}, \tag{3}$$

where,

$$\begin{aligned}\Sigma_{ij} = {} & \nabla_i\nabla_j f_R - g_{ij}\Box f_R + 2R\nabla_i\nabla_j f_G \\ & - 2g_{ij}\Box f\, f_G - 4R_i^{\lambda}\nabla_\lambda\nabla_j f_G \\ & - 4R_j^{\lambda}\nabla_\lambda\nabla_i f_G + 4\Box_{ij} f_G \\ & + 4g_{ij}R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_G \\ & + 4R_{i\alpha\beta j}\nabla^\alpha\nabla^\beta f_G - \tfrac{1}{2}g_{ij}\big(f_R R + f_G G - f(R,G)\big) + (1-f_R)\left(R_{ij}-\tfrac{1}{2}g_{ij}R\right)\end{aligned} \tag{4}$$

Here, $f_R=\dfrac{\partial f(R,G)}{\partial R}$ and $f_G=\dfrac{\partial f(R,G)}{\partial G}$ gives the partial derivatives of $f(R,G)$ with respect to R and G respectively.

In this paper, we discussed the actual behavior of relativistic hydrodynamic source towards the modified $f(R,G)$ gravity model of the form

$$f(R,G)=f_0R^{\alpha}G^{1-\alpha}=R^{\alpha}G^{\beta}, \tag{6}$$

where $f_0>0$ be any constant.

The General Relativistic Hydrodynamics (GRH) equations consist of the local conservation laws of the matter current density, J^m (the continuity equation) and of the stress-energy tensor, T^{mn} (the Bianchi identities):

$$\nabla_m J^m = 0 , \nabla_m T^{mn} = 0 , \tag{7}$$

where usually ∇_m stands for the covariant derivative associated with the four-dimensional space-time metric g_{mn} . The density current is given by $J^m = \rho u^m$, u^m represents the fluid 4-velocity and ρ the proper rest-mass density.

The stress-energy momentum tensor T_{mn} for a non-perfect (un-magnetized) fluid is defined as

$$T_{mn} = \rho(1+E)u_m u_n + (P - \xi\theta)\hbar_{mn} - 2\eta\sigma_{mn} + q_m u_n + q_n u_m, \tag{8}$$

where E is the specific energy density of the fluid in its rest frame, p is the pressure, and $\hbar_{mn}$ is the spatial projection tensor $\hbar_{mn} = u_m u_n + g_{mn}$. In addition, ξ and η are the shear scalar and bulk viscosities.

The scalar expansion θ describes the convergence or divergence of the fluid world lines and finally, q_{mn} is the energy flux vector. In the following, we will be neglecting non-adiabatic effects, like viscosity and heat transfer, considering the stress-energy tensor to be a perfect fluid

$$T_{mn} = \rho\hbar u_m u_n + p g_{mn}. \tag{9}$$

Equation (9) provides,

$$T_{11} = T_{22} = T_{33} = p , T_{44} = -\rho\hbar, \tag{10}$$

where $\hbar$ is the relativistic specific enthalpy and is defined by

$$\hbar = 1 + E + \frac{p}{\rho}. \tag{11}$$

Now, in order to close the system, the equation of motion and the continuity equation must be supplemented with an Equation of State relating some fundamental thermodynamic quantities. In general, the Equation of State takes the form

$$p = p(\rho, \mathrm{E}). \tag{12}$$

The available Equation of State has become sophisticated enough to take into account the physical and chemical processes such as quantization, molecular interactions, nuclear physics, relativistic effects, etc. However, due to their simplicity, the most widely occupied Equation of State in numerical simulations in astrophysics is the ideal fluid Equation of State,

$$p = (\Theta - 1)\rho\mathrm{E} , \tag{13}$$

where Θ is known as adiabatic index.

The polytropic equation of state (e.g., to build equilibrium stellar models),

$$p = \psi\rho^{\Theta} \approx \psi\rho^{1+\frac{1}{\Sigma}} , \tag{14}$$

here Σ be the polytropic index and ψ is the polytropic constant and the microphysical Equation of State that describe the interior of compact stars at nuclear matter densities have also been developed.

Using equations (13) and (14), equation (11) becomes

$$\hbar = 1 + \Theta\mathrm{E}, \tag{15}$$

And

$$\hbar = 1 + \mathrm{E} + \psi\rho^{\frac{1}{\Sigma}} . \tag{16}$$

Thermodynamical Behavior and Entropy of The Model:

Thermodynamic analysis used to examine a gravitational theory has become a useful technique. Recent developments in AdS (Anti-de Sitter Space)/CFT (Conformal

Field Theory) and black hole thermodynamics both explicitly highlight the significance of these occurrences and clearly imply the profound relationship between thermodynamics and gravity. Applying the first and second laws of thermodynamics to the system, we derive the thermodynamical parameters such as temperature and entropy density of the cosmological model in the thermodynamical aspects of relativistic hydrodynamics in gravity. Recently, Shekh et al. [35] obtained the relations in the form of energy density.

Consider the thermodynamical temperature and entropy density of the model is of the form,

$$\tau = \rho^{\frac{\omega}{1+\omega}}, \tag{17}$$

and

$$s' = (1+\omega)\rho^{\frac{\omega}{1+\omega}}. \tag{18}$$

The thermodynamics of the Universe (entropy) which does not depends on any individual fluids. If depends on the total matter density and isotropic pressure of the fluid. Chirde and Shekh [36] are the authors who have investigated the behavior of accelerating spatially homogeneous and isotropic Friedman-Robertson-walker cosmological model with a different option of barotropic viscous fluid in the framework of some well-known $f(T)$ gravity model by defining some basic thermodynamic aspects such as, thermodynamic temperatures and entropy densities of the model with the help of a power law solution. The remark on the actions of thermodynamic parameters is directly related to the energy density of the Universe. Hence our outcomes in equations (17) and (18) show the same features with the work prepared by the above authors.

Metric, Field Equations and Their Solutions:

We investigate the spatially homogeneous and isotropic Freidman-Robertson-Walker (FRW) line element in the form

$$ds^2 = -dt^2 + a^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \; d\varphi^2\right], \tag{19}$$

where $a(t)$ be the average scale factor, is a function of t.

With $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$. The angles θ and φ are the standard azimuthal and polar angles of spherical coordinates, and (t, r, θ, φ) are co-moving coordinates. This indicates that, because of the above coordinate system's homogeneity, the coordinate system expands with space, maintaining the same space coordinates of objects that do not move with regard to the backdrop. In addition, the constant k denotes the space-time curvature. A closed world is represented by $k = 1$, an open universe is represented by $k = -1$ and a flat universe is achieved for $k = 0$. In this work, we discuss the flat universe with infinite radius chosen after $k = 0$.

For the spatially homogeneous and isotropic FRW line element (19) with the fluid of stress-energy tensor, the equation of motion (3) is able to be expressed as

$$2f_R\frac{\dot{a}^2}{a^2} + 3\dot{f}_R\frac{\dot{a}}{a} + 12\dot{f}_R\frac{\dot{a}^3}{a^3} - \frac{1}{2}(Rf_R + Gf_G - f) = -\rho h, \tag{20}$$

$$\ddot{f}_R + 2\dot{f}_R\frac{\dot{a}}{a} + 4\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}\ddot{f}_G + \frac{2\ddot{a}}{a}\dot{f}_G\right) + f_R\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) - \frac{1}{2}(Rf_R + Gf_G - f) = p. \tag{21}$$

The differentiating with respect to time t is shown by the overhead dot.

In order to derive an explicit solution for the system We require an extra constraint, for the same we are considering a hybrid expansion law.

$$a = a_0 \left(\frac{t}{t_0}\right)^{\lambda_1} e^{\lambda_2\left(\frac{t}{t_0}-1\right)}, \tag{22}$$

where a_0 and t_0 stand for the current value of scale factor and universe age respectively, and λ_1, λ_2 are non-negative constants. It is found that while $\lambda_2 = 0$ yields power law cosmology while $\lambda_1 = 0$ yields de-Sitter cosmology.

Kinematical Parameters of the Model:

The special volume is determined by using the value of scale factor given is the equation (22) as

$$V = a^3 = a_0^3 \left(\frac{t}{t_0}\right)^{3\lambda_1} e^{3\lambda_2\left(\frac{t}{t_0}-1\right)}, \tag{23}$$

Form figure 1, it is observed that the behavior of spatial volume as a function of t and reveals that the spatial volume of the universe beginning with constant value as $t \to 0$ and increases as cosmic time tends to infinity. It means that inflation is possible in flat FRW universe. This indicates that the universe start evolving with zero volume and expands with cosmic time t.

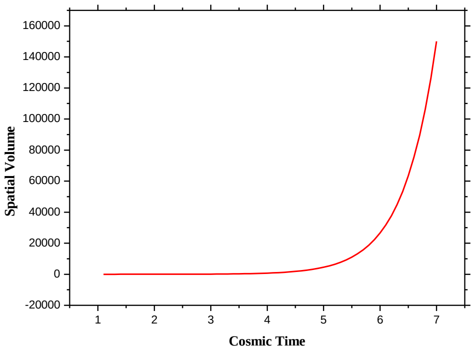

Figure 1: Variation of spatial volume of general relativistic hydrodynamics of isotropic $f(R,G)$ gravity model against time for the appropriate choice of constants $a_0 = 1, t_0 = 1, \lambda_1 = 0.5, \lambda_2 = 0.5$.

The generalized Hubble's parameter and scalar expansion are respectively given by

$$H = \frac{\dot{a}}{a} = \frac{\lambda_1}{t} + \frac{\lambda_2}{t_0}, \tag{24}$$

$$\theta = 3H = 3\left(\frac{\lambda_1}{t} + \frac{\lambda_2}{t_0}\right). \tag{25}$$

From figure 2, Shows that the variations of Hubble's parameter and scalar expansion against time t. In this case at $t \to 0$ the Hubble's parameter and scalar expansion both are constants. When the cosmic time increases, they are approaches to zero. This suggested that at initial stage of the universe, the expansion of the model is much faster and then slow down for later cosmic time this shows that the evolution of the universe starts with finite constant rate and with the expansion it declines.

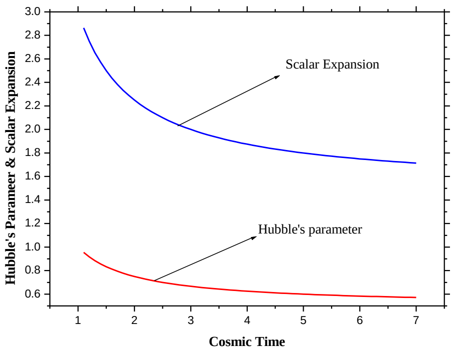

Figure 2: Behavior of Hubble's parameter and scalar expansion of general relativistic hydrodynamic $f(R,G)$ gravity model versus time for the appropriate choice of constants $a_0 = 1, t_0 = 1, \lambda_1 = 0.5, \lambda_2 = 0.5$.

Physical Parameters of the Model:

The Ricci scalar (R) and Guass-Bonnet invariant (G) respectively can be calculate as

$$R = 6\left\{\frac{\lambda_1}{t^2}(2\lambda_1 - 1) + 4\frac{\lambda_1\lambda_2}{tt_0} + 2\frac{\lambda_2^2}{t_0^2}\right\}, \tag{26}$$

$$G = 24\left\{\frac{\lambda_1^3}{t^4}(\lambda_1 - 1) + 2\frac{\lambda_1^2\lambda_2}{t^3 t_0}(2\lambda_1 - 1) + \frac{\lambda_1\lambda_2^2}{t^2 t_0^2}(6\lambda_1 - 1) + \frac{\lambda_2^3}{t_0^3}\left(4\frac{\lambda_1}{t} + \frac{\lambda_2}{t_0}\right)\right\}, \tag{27}$$

From figure 3, shows that the variation of curvature versus time and its behavior is positive over all cosmic time hence our model is stable.

Energy density of the model is

$$\rho = 6^{\alpha}24^{\beta}\tau^{\alpha}\eta^{\beta}\left\{\frac{\alpha+\beta-1}{2} - \frac{\alpha}{3\tau}\frac{\dot{a}^2}{a^2} - \frac{1}{2\tau}\left(M\frac{\dot{\tau}}{\tau} + \alpha\beta\frac{\dot{\eta}}{\eta}\right)\frac{\dot{a}}{a} - \frac{1}{2}\left(\alpha\beta\frac{\dot{\tau}}{\tau^2} + N\frac{\dot{\eta}}{\eta^2}\right)\frac{\dot{a}^3}{a^3}\right\}, \tag{28}$$

where $M = \alpha(\alpha-1)$, $N = \beta(\beta-1)$,

$\tau = \frac{\lambda_1}{t^2}(2\lambda_1 - 1) + \frac{4\lambda_1\lambda_2}{tt_0} + \frac{2\lambda_2^2}{t_0^2}$ and

$$\eta = \frac{\lambda_1^3}{t^4}(\lambda_1 - 1) + \frac{2\lambda_1^2\lambda_2}{t^3t_0}(2\lambda_1 - 1) + \frac{\lambda_1\lambda_2^2}{t^2t_0^2}(6\lambda_1 - 1) + \frac{\lambda_2^3}{t_0^3}\left(\frac{4\lambda_1}{t} + \frac{\lambda_2}{t_0}\right).$$

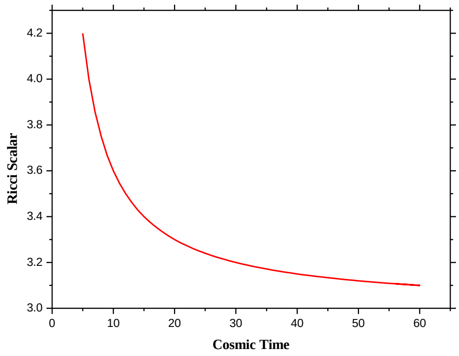

Figure 3: Behavior of Ricci Scalar (Curvature) of general relativistic hydrodynamic $f(R,G)$ gravity model versus time for the appropriate choice of constants $t_0 = 1, \lambda_1 = 0.5, \lambda_2 = 0.5$.

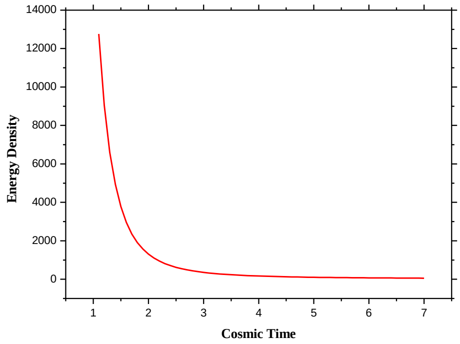

Figure 4: Behavior of Energy density of general relativistic hydrodynamic $f(R,G)$ gravity model versus time for the appropriate choice of constants $a_0 = 1$, $t_0 = 1$, $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\alpha = 1.5$, $\beta = 2$.

Form figure 4, represents the graphical behavior of the energy density against cosmic time with suitable constants. Here we observed that at the initial stage the density of the model continuously decreasing and at the infinity time i.e. $t \to \infty$ then density tends to zero. It means that $\rho \to 0$ as $t \to \infty$, thus our model is asymptotically empty.

Isotropy pressure of the model

$$p = 6^{\alpha}24^{\beta}\tau^{\alpha}\eta^{\beta}\left\{\frac{\alpha+\beta-1}{2} - \frac{1}{6\tau}\left[\begin{matrix} M(\alpha-2)\frac{\dot{\tau}^2}{\tau^2} + M\frac{\ddot{\tau}}{\tau} + 2M\beta\frac{\dot{\tau}\dot{\eta}}{\tau\eta} + N\alpha\frac{\dot{\eta}^2}{\eta^2} \\ +\alpha\beta\frac{\ddot{\eta}}{\eta} + 2\left(M\frac{\dot{\tau}}{\tau} + \alpha\beta\frac{\dot{\eta}}{\eta}\right)\frac{\dot{a}}{a} + \frac{2\dot{a}\ddot{a}}{a^2}\left(\alpha\beta\frac{\dot{\tau}}{\tau} + N\frac{\tau\dot{\eta}}{\eta^2}\right) \\ +\alpha\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + \frac{\dot{a}^2}{a^2}\begin{pmatrix} M\beta\frac{\dot{\tau}^2}{\tau\eta} + \alpha\beta\frac{\ddot{\tau}}{\eta} + 2N\alpha\frac{\dot{\tau}\dot{\eta}}{\eta^2} \\ +N(\beta-2)\frac{\tau\dot{\eta}^2}{\eta^3} + N\frac{\tau\ddot{\eta}}{\eta^2} \end{pmatrix} \end{matrix}\right]\right\}, \tag{29}$$

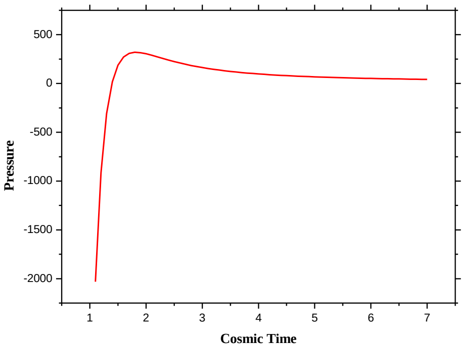

Figure 5: Behavior of Pressure of general relativistic hydrodynamic $f(R,G)$ gravity model versus time for the appropriate choice of constants $a_0 = 1$, $t_0 = 1$, $\lambda_1 = 0.5$, $\lambda_2 = 0.5$, $\alpha = 1.5$, $\beta = 2$.

Figure 5, represents the behavior of pressure of the model against the cosmic time. It shows that at initially when universe start to expand for $0 \leq t < 1.4$ an isotropic pressure is negative while with expansion it becomes positive when $t \geq 1.4$. Hence the behavior of model is accelerated.

The Equation of State

$$\omega=\frac{\left\{\frac{\alpha+\beta-1}{2}-\frac{1}{6\tau}\left[\begin{array}{l}M(\alpha-2)\frac{\dot\tau^2}{\tau^2}+M\frac{\ddot\tau}{\tau}+2M\beta\frac{\dot\tau\dot\eta}{\tau\eta}+N\alpha\frac{\dot\eta^2}{\eta^2}+\alpha\beta\frac{\ddot\eta}{\eta}\\+2\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}+\frac{2\dot a\ddot a}{a^2}\left(\alpha\beta\frac{\dot\tau}{\tau}+N\frac{\tau\dot\eta}{\eta^2}\right)+\alpha\left(\frac{2\ddot a}{a}+\frac{\dot a^2}{a^2}\right)\\+\frac{\dot a^2}{a^2}\left(M\beta\frac{\dot\tau^2}{\tau\eta}+\alpha\beta\frac{\ddot\tau}{\eta}+2N\alpha\frac{\dot\tau\dot\eta}{\eta^2}+N(\beta-2)\frac{\tau\dot\eta^2}{\eta^3}+N\frac{\tau\ddot\eta}{\eta^2}\right)\end{array}\right]\right\}}{\left\{\frac{\alpha+\beta-1}{2}-\frac{\alpha\,\dot a^2}{3\tau a^2}-\frac{1}{2\tau}\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}-\frac{1}{2}\left(\alpha\beta\frac{\dot\tau}{\tau^2}+N\frac{\dot\eta}{\eta^2}\right)\frac{\dot a^3}{a^3}\right\}},\qquad(30)$$

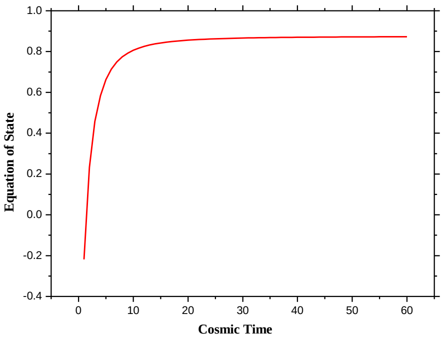

Figure 6: Behavior of equation of state of general relativistic hydrodynamic $f(R,G)$ gravity model versus time for the appropriate choice of constants $a_0=1, t_0=1, \lambda_1=0.5, \lambda_2=0.5, \alpha=1.5, \beta=2$.

Figure 6, depicted the variation of equation of state parameter is a function of t, at initially $t=0$ equation of state parameter having negative value but upper than -1 i.e. the model represent Quintessence model while at $t=1.5$ and at $t>1.5$, then the value of equation of state parameter is positive hence the universe filled with real matter.

Thermodynamical Parameters of the Model:

The thermodynamic temperature of the model is

$$\tau_1=\left[6^\alpha 24^\beta\tau^\alpha\eta^\beta\left\{\begin{array}{l}\frac{\alpha+\beta-1}{2}-\frac{\alpha}{3\tau}\frac{\dot a^2}{a^2}-\frac{1}{2\tau}\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}\\-\frac{1}{2}\left(\alpha\beta\frac{\dot\tau}{\tau^2}+N\frac{\dot\eta}{\eta^2}\right)\frac{\dot a^3}{a^3}\end{array}\right\}\right]^{\varphi(t)}.\qquad(31)$$

where

$$\varphi(t)=\left[\frac{\dfrac{\left\{\frac{\alpha+\beta-1}{2}-\frac{1}{6\tau}\left[\begin{array}{l}M(\alpha-2)\frac{\dot\tau^2}{\tau^2}+M\frac{\ddot\tau}{\tau}+2M\beta\frac{\dot\tau\dot\eta}{\tau\eta}+N\alpha\frac{\dot\eta^2}{\eta^2}+\alpha\beta\frac{\ddot\eta}{\eta}\\+2\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}+\frac{2\dot a\ddot a}{a^2}\left(\alpha\beta\frac{\dot\tau}{\tau}+N\frac{\tau\dot\eta}{\eta^2}\right)+\alpha\left(\frac{2\ddot a}{a}+\frac{\dot a^2}{a^2}\right)\\+\frac{\dot a^2}{a^2}\left(M\beta\frac{\dot\tau^2}{\tau\eta}+\alpha\beta\frac{\ddot\tau}{\eta}+2N\alpha\frac{\dot\tau\dot\eta}{\eta^2}+N(\beta-2)\frac{\tau\dot\eta^2}{\eta^3}+N\frac{\tau\ddot\eta}{\eta^2}\right)\end{array}\right]\right\}}{\left\{\frac{\alpha+\beta-1}{2}-\frac{\alpha\,\dot a^2}{3\tau a^2}-\frac{1}{2\tau}\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}-\frac{1}{2}\left(\alpha\beta\frac{\dot\tau}{\tau^2}+N\frac{\dot\eta}{\eta^2}\right)\frac{\dot a^3}{a^3}\right\}}}{1+\dfrac{\left\{\frac{\alpha+\beta-1}{2}-\frac{1}{6\tau}\left[\begin{array}{l}M(\alpha-2)\frac{\dot\tau^2}{\tau^2}+M\frac{\ddot\tau}{\tau}+2M\beta\frac{\dot\tau\dot\eta}{\tau\eta}+N\alpha\frac{\dot\eta^2}{\eta^2}+\alpha\beta\frac{\ddot\eta}{\eta}\\+2\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}+\frac{2\dot a\ddot a}{a^2}\left(\alpha\beta\frac{\dot\tau}{\tau}+N\frac{\tau\dot\eta}{\eta^2}\right)+\alpha\left(\frac{2\ddot a}{a}+\frac{\dot a^2}{a^2}\right)\\+\frac{\dot a^2}{a^2}\left(M\beta\frac{\dot\tau^2}{\tau\eta}+\alpha\beta\frac{\ddot\tau}{\eta}+2N\alpha\frac{\dot\tau\dot\eta}{\eta^2}+N(\beta-2)\frac{\tau\dot\eta^2}{\eta^3}+N\frac{\tau\ddot\eta}{\eta^2}\right)\end{array}\right]\right\}}{\left\{\frac{\alpha+\beta-1}{2}-\frac{\alpha\,\dot a^2}{3\tau a^2}-\frac{1}{2\tau}\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}-\frac{1}{2}\left(\alpha\beta\frac{\dot\tau}{\tau^2}+N\frac{\dot\eta}{\eta^2}\right)\frac{\dot a^3}{a^3}\right\}}}\right],\qquad(32)$$

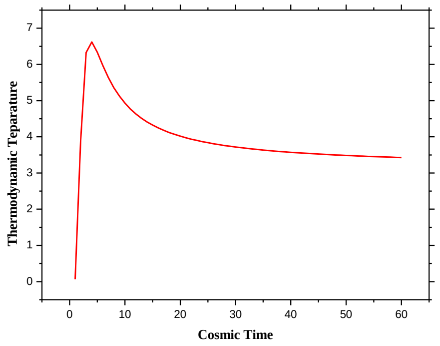

Figure 7: Behavior of Thermodynamic temperature of general relativistic of isotropic model $f(R,G)$ gravity versus time for the appropriate choice of constants $a_0=1, t_0=1, \lambda_1=0.5, \lambda_2=0.5, \alpha=1.5, \beta=2$.

Entropy density of the model

$$s'=\left[1+\frac{\left\{\frac{\alpha+\beta-1}{2}-\frac{1}{6\tau}\left[\begin{array}{l}M(\alpha-2)\frac{\dot\tau^2}{\tau^2}+M\frac{\ddot\tau}{\tau}+2M\beta\frac{\dot\tau\dot\eta}{\tau\eta}+N\alpha\frac{\dot\eta^2}{\eta^2}+\alpha\beta\frac{\ddot\eta}{\eta}\\+2\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}+\frac{2\dot a\ddot a}{a^2}\left(\alpha\beta\frac{\dot\tau}{\tau}+N\frac{\tau\dot\eta}{\eta^2}\right)+\alpha\left(\frac{2\ddot a}{a}+\frac{\dot a^2}{a^2}\right)\\+\frac{\dot a^2}{a^2}\left(M\beta\frac{\dot\tau^2}{\tau\eta}+\alpha\beta\frac{\ddot\tau}{\eta}+2N\alpha\frac{\dot\tau\dot\eta}{\eta^2}+N(\beta-2)\frac{\tau\dot\eta^2}{\eta^3}+N\frac{\tau\ddot\eta}{\eta^2}\right)\end{array}\right]\right\}}{\left\{\frac{\alpha+\beta-1}{2}-\frac{\alpha}{3\tau}\frac{\dot a^2}{a^2}-\frac{1}{2\tau}\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}-\frac{1}{2}\left(\alpha\beta\frac{\dot\tau}{\tau^2}+N\frac{\dot\eta}{\eta^2}\right)\frac{\dot a^3}{a^3}\right\}}\right]$$

$$\times\left[6^\alpha 24^\beta\tau^\alpha\eta^\beta\left\{\begin{array}{c}\frac{\alpha+\beta-1}{2}-\frac{\alpha}{3\tau}\frac{\dot a^2}{a^2}\\-\frac{1}{2\tau}\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}\\-\frac{1}{2}\left(\alpha\beta\frac{\dot\tau}{\tau^2}+N\frac{\dot\eta}{\eta^2}\right)\frac{\dot a^3}{a^3}\end{array}\right\}\right]^{\psi(t)},\qquad(33)$$

where

$$\psi(t)=\left[1+\frac{\left\{\frac{\alpha+\beta-1}{2}-\frac{1}{6\tau}\left[\begin{array}{l}M(\alpha-2)\frac{\dot\tau^2}{\tau^2}+M\frac{\ddot\tau}{\tau}+2M\beta\frac{\dot\tau\dot\eta}{\tau\eta}+N\alpha\frac{\dot\eta^2}{\eta^2}+\alpha\beta\frac{\ddot\eta}{\eta}\\+2\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}+\frac{2\dot a\ddot a}{a^2}\left(\alpha\beta\frac{\dot\tau}{\tau}+N\frac{\tau\dot\eta}{\eta^2}\right)+\alpha\left(\frac{2\ddot a}{a}+\frac{\dot a^2}{a^2}\right)\\+\frac{\dot a^2}{a^2}\left(M\beta\frac{\dot\tau^2}{\tau\eta}+\alpha\beta\frac{\ddot\tau}{\eta}+2N\alpha\frac{\dot\tau\dot\eta}{\eta^2}+N(\beta-2)\frac{\tau\dot\eta^2}{\eta^3}+N\frac{\tau\ddot\eta}{\eta^2}\right)\end{array}\right]\right\}}{\left\{\frac{\alpha+\beta-1}{2}-\frac{\alpha\,\dot a^2}{3\tau a^2}-\frac{1}{2\tau}\left(M\frac{\dot\tau}{\tau}+\alpha\beta\frac{\dot\eta}{\eta}\right)\frac{\dot a}{a}-\frac{1}{2}\left(\alpha\beta\frac{\dot\tau}{\tau^2}+N\frac{\dot\eta}{\eta^2}\right)\frac{\dot a^3}{a^3}\right\}}\right]^{-1}.\qquad(34)$$

In figure 7, It is observed that the variation of thermodynamic temperature against cosmic time t. Initially the temperature increasing but after some time it goes on decrease then constant at large time whereas figure 8, represents the behavior of entropy density of the model versus time. It is observed that the behavior of entropy density against time t. As the energy density and entropy density both are directly related to each other with the power term of equation of state parameter of the model. It shows that the thermodynamic entropy density as similar to the energy density of the model. At cosmic time tends to infinity the thermodynamic density tends to zero.

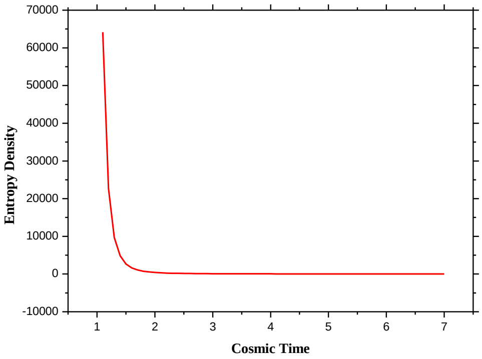

Figure 8: Behavior of Entropy density of general relativistic of isotropic model $f(R,G)$ gravity versus time for the appropriate choice of constants $a_0 = 1, t_0 = 1, \lambda_1 = 0.5, \lambda_2 = 0.5, \alpha = 1.5, \beta = 2$
.

Conclusion:

In the analysis of isotropic homogeneous FRW cosmological model with modified form of $f(R,G)$ gravity with the relativistic hydrodynamic fluid, the behavior of spatial volume as a function of t and reveals that the spatial volume of the universe is beginning with constant value as $t \to 0$ and increases as cosmic time tends to infinity. It means that inflation is possible in flat FRW universe. This indicates that the universe start evolving with zero volume and expands with cosmic time t along with the variations of Hubble's parameter and scalar expansion against time t. In this case at $t \to 0$ the Hubble's parameter and scalar expansion both are constants. When the cosmic time increases, they are approaches to zero. This suggested that at initial stage of the universe, the expansion of the model is much faster and then slow down for later cosmic time this shows that the evolution of the universe starts with finite constant rate and with the expansion it declines. The curvature versus time and its behavior is positive over all cosmic time hence our model is stable (see figure 3).

At the initial stage the density of the model continuously decreasing and at the infinity time i.e. $t \to \infty$ then density tends to zero. It means that $\rho \to 0$ as $t \to \infty$, thus our model is asymptotically empty (refer figure 4) also, the behavior of pressure of the model against the cosmic time shows at initially when universe start to expand for $0 \le t < 1.4$ an isotropic pressure is negative while with expansion it becomes positive when $t \ge 1.4$. Hence the behavior of model is accelerated. From figure 6, the variation of equation of state parameter is a function of t, at initially $t = 0$ equation of state parameter having negative value but upper than -1 i.e. the model represents Quintessence model while at $t = 1.5$ and at $t > 1.5$ then the value of equation of state parameter is positive hence the universe filled with real matter.

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