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A Critical Survey on Certain Recent Inequalities Involving Trigonometric and Hyperbolic Functions

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Abstract:

Analytical inequalities serve as a cornerstone in the foundation of mathematics, providing deep understandings into the relationships between various mathematical functions. In this comprehensive survey paper, we discuss on a series of notable research contributions that generalize fundamental inequalities in mathematics and explore their applications in diverse domains. The inequalities corroborated by Dhaigude, Bagul, Thool, Chesneau, Kostić, in some selected papers only, are central to our survey paper. Their collaborated work has helped us to understand different sides of inequalities and how they can be useful in further research.

Keywords: *Sinc and hyperbolic sinc functions; Circular and inverse circular functions; Hyperbolic and inverse hyperbolic functions; Exponential function; Jordan's inequality; Cusa-Huygen's inequality.*

2020 Mathematics Subject Classification: 26D05; 26D07; 26D10; 26D99; 33B10.

1. Introduction:

We start by looking towards analytical inequalities and introducing the authors whose findings are the foundation of this study. Within this paper, we discussed about the development and generalizations of fundamental inequalities having algebraic and exponential bounds. We try to explore the mathematical inequalities along with their significance. We aim to highlight the richness of analytical inequalities, which significantly enhanced our understanding of these mathematical tools. In this paper, we have highlighted the latest developments about analytical inequalities, aims to inspire future researchers to continue.

2. Lemma:

This lemma [1, Theorem.1.25] is due to Anderson et al., which plays a vital role in the theory of approximations and analysis. It is very much useful and widely used by researchers in respect of analytical inequalities, termed as l'Hôpital's rule of monotonicity and can be stated as follows:

Let $f, g: [a, b] \to \mathbb{R}$ be two continuous functions on $[a, b]$ and differentiable on (a, b) s.t $g'(t) \neq 0, \forall\, t \in (a\ b)$. If f'/g' is increasing ↑ (or decreasing ↓) on (a, b), then the functions $A(t) = \frac{f(t)-f(a)}{g(t)-g(a)}$ and $B(t) = \frac{f(t)-f(b)}{g(t)-g(b)}$ are also increasing ↑ (or decreasing ↓) on (a, b). The strictness of the monotonicity of $A(t)$ and $B(t)$ depends on behavior of $f'(t)/g'(t)$.

3. Literature Review:

For this paper we have taken into consideration some important and relevant references [2-18], mainly focusing on analytical inequalities by Bagul et al. [2-14, 16, 18], Thool et al. [3, 4, 6, 15, 18]. They established inequalities for trigonometric functions, hyperbolic functions and for their quotients along with the inverses. Such inequalities were deeply discussed in [2-18] and references therein.

We have discussed comprehensive review, in the sections *3.1* to *3.5*, whose graphical conclusions along with justification are provided in the references. In section 3.1, we have discussed about the inequalities involving circular and inverse circular functions. Section *3.2* is dedicated to the inequalities involving hyperbolic and inverse hyperbolic functions. $\sin t \sinh t\,;\cos t \cosh t$ Inequalities and ratio functions inequalities are discussed in sections *3.3* and *3.4* respectively. Section *3.5* focuses on Wilker type and Cusa-Huygen's type inequalities. After that we have pointed out some open problems, which will help researcher for further improvement and refinement of analytical inequalities. Also, we have posed a new open problem which includes algebraic and exponential type bounds for $\frac{x}{\tan^{-1} x} + \frac{x}{\tanh^{-1} x}$.

3.1 Circular and Inverse Circular Functions inequalities:

The journey of these functions and their inequalities is a rich and fascinating one, spanning thousands of years. Euclid (fl. 300 BCE) wrote "Elements," which included trigonometric concepts. Indians, viz; Aryabhata (476 CE) developed the concept of zero and worked on trigonometry, Brahmagupta (598 CE) introduced the sine, cosine, and tangent functions. Over the years, many important labeled inequalities like Cauchy-Schwarz's inequality, Jensen's inequality, Hölder-Minkowski's inequality, etc. were studied carefully. Recently Cusa-Huygen's inequality, Wilker's inequality, Jordon's inequality and their hyperbolic versions are refined regularly by mathematicians. In this section, we mainly focused on inequalities and their bounds along with refinements for the circular and inverse circular functions. In 2017, Bagul [11] obtained some new lower and upper bounds for $\cos t$, $\sin t/t$ and established inequalities involving circular, hyperbolic and exponential functions.

The well-known labelled inequalities are given below:

Jordan's inequality: $\frac{2}{\pi} < \frac{\sin t}{t} < 1, t \in (0, \pi/2)$: [5, inequality 1] (3.1.1)

Kober's inequality: $1 - \frac{2t}{\pi} < \cos t < 1 - \frac{t^2}{\pi}$ $t \in (0, \pi/2)$: [5, inequality 2] (3.1.2)

Becker's inequality: $\frac{\pi^2 - 4t^2}{\pi^2} < \frac{t}{\tan t} < \frac{\pi^2 - 4t^2}{8}$, $t \in (0, \pi/2)$: [8, inequality 1.6] (3.1.3)

The above inequalities $(3.1.1), (3.1.2)$ and $(3.1.3)$ were highlighted by authors in [5] and [11]. Reader may refer [3-6, 8, 9, 11, 13, 14, 16] and references therein to study various versions of $(3.1.1) - (3.1.3)$ along with their refinements. Inequality for $\sin t/t$ with algebraic bounds mentioned in [8], see (3.1.4), whose upper bound was then refined in [8, Theorem 2.1], see (3.1.5).

$1 - \frac{t^2}{6} < \frac{\sin t}{t} < 1 - \frac{2t^2}{3\pi^2}$, $t \in (-\pi/2, \pi/2)$: [8, inequality 1.2] (3.1.4)

and

$\left(1 + \frac{2(\pi-2)}{\pi^2} t^2\right)^{-1} < \frac{\sin t}{t} < \left(1 + \frac{t^2}{6}\right)^{-1}$, $t \in (0, \pi/2)$: [8, Theorem 2.1] (3.1.5)

Combining (3.1.4) and (3.1.5), yields (3.1.6) as follows:

$$1-\frac{t^2}{6}<\frac{\sin t}{t}<\left(1+\frac{t^2}{6}\right)^{-1}, t\in\ (0,\pi/2)\text{: [8, inequality 2.3]} \qquad (3.1.6)$$

Note that, the bounds discussed in (3.1.4)-(3.1.6) are rational functions which have less number of algebraic operations, hence these are simple efficient in nature.

The inequality for $t/\tan t$ using rational function is given in [8],

$$\left(1+\frac{\tan\kappa-\kappa}{\kappa^3}t^2\right)^{-1}<\frac{t}{\tan t}<\left(1+\frac{t^2}{3}\right)^{-1}, t\in\ (0,\pi/2)\text{: [8, Theorem 2.3]} \qquad (3.1.7)$$

for some $\kappa\in(0,\pi/2)$, where $\left\{\frac{2(\pi-2)}{\pi^2},\frac{1}{6}\right\}$ and $\left\{\frac{\tan\kappa-\kappa}{\kappa^3},\frac{1}{3}\right\}$ are the sets of best possible constants for inequalities (3.1.5) and (3.1.7) respectively. These two inequalities are then sharpen in [11, 13] by providing exponential bounds and (3.1.7) was improvised in [6] by extending it to larger domain $i.e.\,(0,\pi/2)$, see inequalities (3.1.8) and (3.1.9)

$$e^{\ln(\sin 1)t^2}<\frac{\sin t}{t}<e^{-t^2/6}, t\in\ (0,1)\text{: [13, Theorem 2],} \qquad (3.1.8)$$

and

$$e^{\frac{-\ln\left(\frac{\tan\kappa}{\kappa}\right)}{\kappa^2}t^2}<\frac{t}{\tan t}<e^{-t^2/3}, t\in\ (0,\pi/2)\text{: [6, Theorem 5]} \qquad (3.1.9)$$

Refinement of (3.1.2) with algebraic bounds was mentioned in [5],

$$1-\frac{t^2}{2}<\cos t<1-\frac{4t^2}{\pi^2}, t\in\ (0,\pi/2)\text{: [5, inequality 5]} \qquad (3.1.10)$$

and with exponential bounds was mentioned in [13],

$$e^{\ln(\cos 1)t^2}<\cos t<e^{-t^2/2}, t\in\ (0,1)\text{: [13, Theorem 1]} \qquad (3.1.11)$$

for same κ that was defined in (3.1.7), where $\left\{-\ln(\sin 1),\frac{1}{6}\right\}$, $\left\{\kappa^{-2}\ln\left(\frac{\tan\kappa}{\kappa}\right),\frac{1}{3}\right\}$ and $\left\{-\ln(\cos 1),\frac{1}{2}\right\}$ are the sets of best possible constants for inequalities (3.1.8), (3.1.9) and (3.1.11) respectively. Also, (3.1.9) was improved in [4] as follows:

$$\frac{t}{\tanh t}-\left(\frac{2\coth(\pi/2)}{\pi}\right)t^2<\frac{t}{\tan t}<\frac{t}{\tanh t}-\frac{2}{3}t^2, t\in\ (0,\pi/2)\text{: [4, Lemma 2]} \qquad (3.1.12)$$

In an attempt to obtain refinement for existing inequalities, mathematicians tried various combinations of functions for their tails. Now, we present such inequalities as follows:

$$\frac{1+2\cos t}{2+\cos t}<\frac{\sin t}{t}<\sqrt{\frac{(t+\sin t)(1+\cos t)}{4t}}, t\in(0,\pi) \qquad (3.1.13)$$

Left hand side inequality of (3.1.13) is due to [7, inequality 3.6] and right hand side inequality is due to [3, inequality 2.6], in which upper and lower bounds are combinations of algebraic and trigonometric functions. One of such inequality was proved by Dhaigude et al. [16] with the help of simple derivative technique,

$$\frac{4+\cos t}{5}-\frac{t^2}{15}<\frac{\sin t}{t}, t\in(0,\pi)\text{: [16, Theorem 2.1]} \qquad (3.1.14)$$

In 2021, Bagul et al.[5] established inequalities with polynomial-exponential bounds for $\sin t/t$ and $\cos t$ as follows:

$$\left(1-\frac{t^2}{\pi^2}\right)\exp\left(-\frac{\ln 2}{\pi^2}t^2\right)<\frac{\sin t}{t}<\left(1-\frac{t^2}{\pi^2}\right)\exp\left(\left(\frac{1}{\pi^2}-\frac{1}{6}\right)t^2\right), t\in(0,\pi) \qquad (3.1.15)$$

and

$$\left(1-\frac{4t^2}{\pi^2}\right)\exp\left(\frac{4\ln(\pi/4)}{\pi^2}t^2\right)<\cos t<\left(1-\frac{4t^2}{\pi^2}\right)\exp\left(\left(\frac{4}{\pi^2}-\frac{1}{2}\right)t^2\right), t\in(0,\pi/2)$$

(3.1.16)

where $\left\{-\frac{\ln 2}{\pi^2},\frac{1}{\pi^2}-\frac{1}{6}\right\}$ and $\left\{\frac{4\ln(\pi/4)}{\pi^2},\frac{4}{\pi^2}-\frac{1}{2}\right\}$ are the sets of best possible constants for inequalities (3.1.15) and (3.1.16) respectively. These were proved with the aid of power series expansions of certain functions and *l'Hôpital's rule of monotonicity*(LRM) along with even indexed Bernoulli numbers. Another inequality was devised in [5] as an application to (3.1.15) and (3.1.16) as follows:

$$\phi_1(t) < \cos t < \left(1-\frac{4t^2}{\pi^2}\right)\exp\left(\left(\frac{4}{\pi^2}-\frac{1}{2}\right)t^2\right), t \in (0,\pi/2)\text{: [5, Proposition 1]} \qquad (3.1.17)$$

where $\phi_1(t) = 1-\frac{3\pi^2(\pi^2-12)}{(\pi^2-6)^2}+\frac{3\pi^2}{(\pi^2-6)^2}\left[(\pi^2-6)\left(1-\frac{t^2}{\pi^2}\right)-6\right]\exp\left(\left(\frac{1}{\pi^2}-\frac{1}{6}\right)t^2\right)$.

In [3] Bagul et al. attempted to obtain circular-exponential type bounds for $\sin t/t$, they obtained several bounds for it, we present here some special cases only as partial part of Corollary 1 and Corollary 2 in [3],

$$\cos\left(\frac{t}{\sqrt{3}}\right) < \frac{\sin t}{t} < \cos\left(\frac{t}{\sqrt{3}}\right)e^{\alpha_1 t^2}, t \in (0,\pi/2)\text{: [3, Corollary 1]} \qquad (3.1.18)$$

and

$$\cos\left(\frac{t}{2}\right)e^{\alpha_2 t^2} < \frac{\sin t}{t} < \cos\left(\frac{t}{\sqrt{3}}\right)e^{-t^2/24}, t \in (0,\pi/2)\text{: [3, Corollary 2]} \qquad (3.1.19)$$

where $\alpha_1 = \frac{4}{\pi^2}\ln\left\{2/\pi\, cos\left(\frac{\pi}{2\sqrt{3}}\right)\right\}$ and $\alpha_2 = \frac{4}{\pi^2}\ln\left\{\frac{2\sqrt{2}}{\pi}\right\}$ and here $\{0,\alpha_1\}$ and $\left\{\alpha_2, -\frac{1}{24}\right\}$ are the sets of best possible constants for inequalities (3.1.18) and (3.1.19) respectively such that those inequalities hold true..

Now we discuss, the inequalities having bounds of exponential type, where the base will be combination of algebraic functions, exponential functions and hyperbolic functions. In [9], authors have given three inequalities as follows:

For $t \in (0,\pi/2)$,

$$\left(\frac{\tanh t}{t}\right)^{\alpha_3} < \frac{\sin t}{t} < \left(\frac{\tanh t}{t}\right)^{1/2}\text{ : [3, Proposition 1.2]} \qquad (3.1.20)$$

$$\left(\frac{2+e^{-t^2}}{3}\right)^{\alpha_4} < \frac{\sin t}{t} < \left(\frac{2+e^{-t^2}}{3}\right)^{1/2}\text{ : [3, Proposition 1.3]} \qquad (3.1.21)$$

and

$$\left(\frac{3}{2+\cosh t}\right)^{\alpha_5} < \frac{\sin t}{t} < \left(\frac{3}{2+\cosh t}\right)^{1}\text{ : [3, Proposition 1.3]} \qquad (3.1.22)$$

where $\alpha_3 = \ln\left(\frac{2}{\pi}\right)/\ln\left(\frac{2\, tanh\left(\frac{\pi}{2}\right)}{\pi}\right)$, $\alpha_4 = \ln\left(\frac{2}{\pi}\right)/\{\ln\left(2+e^{-(\pi/2)^2}\right)-\ln 3\}$, and

$\alpha_5 = -\ln\left(\frac{2}{\pi}\right)/\{\ln(2+\cosh(\pi/2))-\ln 3\}$. Here, $\left\{\alpha_3,\frac{1}{2}\right\}$, $\left\{\alpha_4,\frac{1}{2}\right\}$ and $\{\alpha_5,1\}$ are the sets of best possible constants such that (3.1.20)-(3.1.22) hold true over the natural domain of interval. (3.1.20)-(3.1.21) were proved with the help of function $H(t) = \frac{\sin t - tcost}{t^2 \sin t}$, which is strictly positive increasing on $(0,\pi/2)$ and (3.1.22) was proved using (3.1.8). One can observe that (3.1.22) is Cusa-Huygen's type inequality.

Dhaigude et al.[16] provided double inequality which is generalization of (3.1.14) and it is stated below in (3.1.23) for $m = 2n-1, n \in \mathbb{N}\ and\ t \in (0,\pi)$

$$\phi_2(t) < \frac{\sin t}{t} < \phi_3(t)\text{ : [16, Theorem 2.2]} \qquad (3.1.23)$$

where

$$\phi_2(t) = \frac{(2m+2)+\cos t}{2m+3}+\frac{2}{2m+3}\sum_{k=1}^{m+1}\frac{k-m-1}{(2k+1)!}(-1)^{k+1}t^{2k}$$

and

$$\phi_3(t) = \frac{(2m+4)+\cos t}{2m+5}+\frac{2}{2m+5}\sum_{k=1}^{m+2}\frac{k-m-2}{(2k+1)!}(-1)^{k+1}t^{2k}$$

(3.1.23) is not only refinement of (3.1.14), but also of all the inequalities presented in this survey paper for $\sin t/t$, it was proved by making smart use of power series expansions of circular functions $\sin t/t$, $\cos t$ with an aid of Leibnitz's rule for the convergence of an infinite alternating series. Authors in [16] have discussed some particular cases of (3.1.23), from which one can observe that the sharpness of upper and lower bound increases as n increases $i.e.$ if $n \to \infty$, then $\left|\frac{\sin t}{t} - \phi_i(t)\right| \to 0, for\ i = 1, 2$. Moreover, the double sided inequalities have been discussed by Bagul & Chesneau [10] in 2019 for trigonometric and hyperbolic functions, along with open problems that will be presented in separate section. The double sided inequalities were displayed as (3.1.24) and (3.1.25)

$$2 + \cos^2(t) > \frac{\sin(2t)}{2t} + \frac{2\sin(t)}{t} > 1 + 2\cos(t), t \in (0, \pi/2)\text{: [10, Theorem 2.1]} \qquad (3.1.24)$$

Equivalently (3.1.24) can be written as

$$\frac{2}{\cos(t)} + \cos(t) > \frac{\sin(t)}{t} + \frac{2\tan(t)}{t} > \frac{1}{\cos(t)} + 2,\ t \in (0, \pi/2)\text{: [10, Remark 1]} \qquad (3.1.25)$$

Inverse circular functions have same importance as that of the circular functions. Their inequalities with algebraic bounds have been highlighted by authors of [15], similar inequalities can be studied by referring the references therein.

$$1 + \frac{t^2}{6} < \frac{\sin^{-1} t}{t} < 1 + \left(\frac{\pi-2}{2}\right) t^2, t \in (0,1)\text{: [15, inequality 1.3]} \qquad (3.1.26)$$

and rational function type bounds can be seen in

$$\frac{6}{6-t^2} < \frac{\sin^{-1} t}{t} < \frac{\pi}{\pi-(\pi-2)t^2}, t \in (0,1)\text{: [15, inequality 1.4]} \qquad (3.1.27)$$

Exponential type bounds was given in [11],

$$e^{t^2/6} < \frac{\sin^{-1} t}{t} < e^{\ln(\pi/2)t^2},\ t \in (0,1)\text{: [11, Theorem 1]} \qquad (3.1.28)$$

The upper and lower bounds in (3.1.28) are sharper than that of (3.1.26), whereas (3.1.27) is complete refinement of (3.1.26) and (3.1.28). Exponential and circular type bounds for $\tan^{-1} t / t$ have been discussed in [6, 11]. In [11], the authors provided inequalities (3.1.28) and (3.1.29) – (3.1.30), since the proofs of different theorems provided in [11] can not be consider flawless, so, the alternative proofs of these theorems presented in [6], also, some inequalities modified by changing their interval of domain.

$$e^{-t^2/3} < \frac{\tan^{-1} t}{t} < 1,\ t \in (0, \infty)\text{: [6, Theorem 2]} \qquad (3.1.29)$$

$$\frac{t}{\tan t} < \frac{\tan^{-1} t}{t}, t \in (0, 1)\text{: [11, Remark 2]} \qquad (3.1.30)$$

Dhaigude et al. [15] made an attempt to refine (3.1.28) by providing polynomial-exponential type bounds as follows:

$$\left(1 - \frac{t^2}{4}\right)^{-1} e^{-t^2/12} < \frac{\sin^{-1} t}{t} < \left(1 - \frac{t^2}{4}\right)^{-1} e^{\ln(3\pi/8)t^2}, t \in (0, 1)\text{: [15, Theorem 2.1]} \qquad (3.1.31)$$

and

$$\left(1 + \frac{t^2}{4}\right)^{-1} e^{5t^2/12} < \frac{\sin^{-1} t}{t} < \left(1 + \frac{t^2}{4}\right)^{-1} e^{\ln(5\pi/8)t^2}, t \in (0, 1)\text{: [15, Theorem 2.2]} \qquad (3.1.32)$$

(3.1.31) is sharper than that of (3.1.32), moreover it forms refinement of (3.1.28). Thool et al. [18] discusses about ratio functions of inverse circular and inverse hyperbolic functions, where we can find mixed function inequalities for both inverse circular and inverse hyperbolic functions. Due to [18, Lemma 7] we will have (3.1.34). For $t \in (0, 1)$, by combining Lemma 3 and Lemma 4 of [18], we write the following inequality,

$$\frac{\sin^{-1} t}{t} < \frac{3}{(3+t^2)\sqrt{1-t^2}} < \left(\frac{\sinh^{-1} t}{t}\right)^2 (1 - t^2)^{-1/2} < \frac{1}{\sqrt{1-t^2}}\text{ : [18, Lemma 3, 4]} \qquad (3.1.33)$$

$$\frac{t}{\tan^{-1} t} + \frac{t}{\tanh^{-1} t} < 2\,, t \in (0,1) : \text{[18, Lemma 7]} \tag{3.1.34}$$

3.2 Hyperbolic and Inverse Hyperbolic Functions inequalities:

These functions introduced in the 18th century by Johann Heinrich Lambert (1728-1777) Swiss mathematician. Extensions to trigonometric functions, they share similar properties. These functions are crucial in solving differential equations, particularly in physics and engineering, where they describe hyperbolic curves and surfaces. Inverse hyperbolic functions were developed meanwhile. Recently, there has been significant work on refining inequalities, including their inverses. Researchers have been actively contributing to this field, focusing on refinements of existing inequalities so far in the literature.

Like labelled inequalities for circular functions, the hyperbolic function's inequalities studied extensively over a period of time. The hyperbolic versions of Jordan's, Kober's, Becker-Stark's, etc inequalities were studied and refined by mathematicians. In [8, 12], authors presented inequalities for hyperbolic functions on unit interval, which were then generalized to find best possible constants such that the corresponding inequalities hold true. For detailed information, one can see closely related references in [8, 12]. The inequalities with polynomial bounds are as follows:

$$1 + \frac{t^2}{6} < \frac{\sinh t}{t} < 1 + \frac{t^2}{5}\,, t \in (0,1) \tag{3.2.1}$$

$$1 + \frac{t^2}{2} < \cosh t < 1 + \frac{5t^2}{9}\,, t \in (0,1) \tag{3.2.2}$$

$$\frac{t}{\tanh t} < 1 + \frac{t^2}{3}\,, t \in (0,1) \tag{3.2.3}$$

These inequalities $(3.2.1) - (3.2.3)$ refined in [8, Theorem 2.2, Theorem 2.4] and are given below:

$$1 + \frac{t^2}{6} < \frac{\sinh t}{t} < 1 + (\sinh 1 - 1)t^2\,, t \in (0,1) \tag{3.2.4}$$

$$1 + \frac{t^2}{2} < \cosh t < 1 + (\cosh 1 - 1)t^2\,, t \in (0,1) \tag{3.2.5}$$

$$1 + \left(\frac{1-\tanh 1}{\tanh 1}\right) t^2 < \frac{t}{\tanh t} < 1 + \frac{t^2}{3}\,, t \in (0,1) \tag{3.2.6}$$

where $\left\{\frac{1}{6}, \sinh 1 - 1\right\}, \left\{\frac{1}{2}, \cosh 1 - 1\right\}, \left\{\frac{1-\tanh 1}{\tanh 1}, \frac{1}{3}\right\}$ are the sets of best possible constants $s.t.$ $(3.2.4) - (3.2.6)$ hold true. By taking reciprocal of these inequalities, we will get rational functions as bounds for functions $\frac{t}{\sinh t}, \frac{1}{\cosh t}, \frac{\tanh t}{t}$. Another such inequality for $\cosh t$ presented by authors of [12],

$$\frac{3}{3-t^2} \le \cosh t \le \frac{2}{2-t^2}\,, t \in (0,1) : \text{[12, inequality 1.6]} \tag{3.2.7}$$

Authors [9, 11-13] established inequalities for $\frac{\sinh t}{t}, \cosh t\,, \frac{\tanh t}{t}$ having exponential functions as their bounds on unit interval,

$$e^{\ln(\sinh 1)t^2} < \frac{\sinh t}{t} < e^{\frac{t^2}{6}}\,, t \in (0,1) : \text{[13, Theorem 3]} \tag{3.2.8}$$

$$e^{\ln(\cosh 1)t^2} < \cosh t < e^{t^2/2}\,, t \in (0,1) : \text{[12, Theorem 2.1]} \tag{3.2.9}$$

$$e^{\ln(\tanh 1)t^2} < \frac{t}{\tanh t} < e^{t^2/3}\,, \ t \in (0,1) : \text{[11, Theorem 5]} \tag{3.2.10}$$

where $\{\ln(\sinh 1), 1/6\}, \{\ln(\cosh 1), 1/2\}, \{\ln(\tanh 1), 1/3\}$ are the sets of best possible constants such that $(3.2.8) - (3.2.10)$ holds good on $(0,1)$. One can easily observe that, the upper and lower bounds of $(3.2.8)$ and $(3.2.9)$ are not comparable with those of $(3.2.4)$ and

(3.2.5) respectively. Here, (3.2.6) is the complete refinement of (3.2.10). In [9, Proposition 1.6, 1.7], they have presented an inequality for $\cosh t$ with special kind of exponential bounds, see (3.2.11) and (3.2.12):

$$\left(1+\frac{t^2}{3}\right)^{3/2} \leq \cosh t \leq \left(1+\frac{t^2}{d_1}\right)^{\frac{d_1}{2}}, t \in (0,1) with\ d_1 \approx 3.194528 \qquad (3.2.11)$$

$$\exp\left(\frac{3}{2}\left(1-e^{-\frac{t^2}{3}}\right)\right) \leq \cosh t \leq \exp\left(\frac{1}{2d_2}\left(1-e^{-d_2t^2}\right)\right), t \in (0,1)\ with\ d_2 \approx 0.272342$$

(3.2.12)

(3.2.11) refined (3.2.9) and (3.2.12) completely, whereas the upper bound of (3.2.12) is tighter than that of (3.2.9). Polynomial-exponential type bounds obtained by Bagul et al.[5] for the hyperbolic functions $\frac{\sinh t}{t}$ and $\cosh t$, these are depicted below:

If r is any non-zero positive real number, then following inequalities are true

$$\left(1+\left(\frac{t}{\pi}\right)^2\right)e^{d_3t^2} < \frac{\sinh t}{t} < \left(1+\left(\frac{t}{\pi}\right)^2\right)e^{d_4t^2}, t \in (0,r) \qquad (3.2.13)$$

$$\left(1+\left(\frac{2t}{\pi}\right)^2\right)e^{d_5t^2} < \cosh t < \left(1+\left(\frac{2t}{\pi}\right)^2\right)e^{d_6t^2}, t \in (0,r) \qquad (3.2.14)$$

where $d_3 = r^{-2}\ln\left[\frac{\pi^2 \sinh r}{r(\pi^2+r^2)}\right]$, $d_4 = \frac{1}{6}-\frac{1}{\pi^2}$, $d_5 = r^{-2}\ln\left[\frac{\pi^2\cosh r}{r(\pi^2+4r^2)}\right]$, $d_6 = \frac{1}{2}-\frac{4}{\pi^2}$ and $\{d_3, d_4\}, \{d_5, d_6\}$ are the sets of best possible constants in concern with (3.2.13), (3.2.14) receptively. (3.2.13) is the complete refinement of (3.2.4) and (3.2.8), whereas (3.2.14) is refinement of (3.2.5), (3.2.9), (3.2.11) and (3.2.12). By integrating (3.2.13), authors [5] obtained very sharp inequality for $\cosh t$ as follows:

$$\psi_1(t) < \cosh t < \psi_2, t \in (0,r) and\ r > 0 : \text{[5, Proposition 2]} \qquad (3.2.15)$$

where $\psi_1(t) = \left(1+\frac{1-d_3\pi^2}{2d_3^2\pi^2}\right)+\frac{1}{2d_3^2\pi^2}\left[d_3\pi\left(1+\left(\frac{t}{\pi}\right)^2\right)-1\right]e^{d_3t^2}$, and

$$\psi_2 = \left(1-\frac{3\pi^4}{(\pi^2-6)^2}\right)+\frac{3\pi^2}{(\pi^2-6)^2}\left[(\pi^2-6)^2\left(1+\left(\frac{t}{\pi}\right)^2\right)-6\right]e^{d_4t^2},$$

Above inequality (3.2.15) is a refinement of all the inequalities presented in this paper, for hyperbolic cosine function. The hyperbolic-exponential bounds for function $\frac{\sinh t}{t}$ obtained in [3, THEOREM 2], see (3.2.16),

$$\cosh\left(\frac{t}{2}\right)e^{d_7t^2} < \frac{\sinh t}{t} < \cosh\left(\frac{t}{2}\right)e^{t^2/24}, 0 < t \leq r, and\ r > 0 \qquad (3.2.16)$$

where $d_7 = r^{-2}\ln\left[\frac{\sinh r}{r\cosh(r/2)}\right]$ and $\frac{1}{24}$ are the best possible constants for (3.2.16). This inequality provides sharp upper and lower bounds than that of (3.2.13). In an attempt to obtain above inequality, authors corroborated sharp upper bound for hyperbolic sinc function, which is depicted below:

$$\frac{\sinh t}{t} < \sqrt{\frac{(t+\sinh t)(1+\cosh t)}{4t}} < \frac{2+\cosh t}{3}, t \in \mathbb{R}\setminus\{0\} \qquad (3.2.17)$$

Though (3.2.13), (3.2.16) and (3.2.17) are incomparable, (3.2.17) provides sharper bounds on $[3.894, \infty)$ and $[4.816, \infty)$ in concern with the inequalities (3.2.13) and (3.2.16) respectively.

In [7, 9] authors have provided Cusa-Huygen's type inequalities for $\frac{\sinh t}{t}$ as a central function are given below:

$$\frac{\sinh t}{t} < \left(\frac{1+\cosh t}{2}\right)^{d_8}, t \in (0,\pi/2) : \text{[7, Corollary 1]} \qquad (3.2.18)$$

$$\left(\frac{2+\cos t}{3}\right)^{d_9} < \frac{t}{\sinh t} < \left(\frac{2+\cos t}{3}\right)^{d_{10}}, t \in (0,\pi/2) : \text{[9, Proposition 1.4]} \qquad (3.2.19)$$

$$\left(\frac{2+\cosh t}{3}\right)^{d_{11}} < \frac{\sinh t}{t} < \left(\frac{2+\cosh t}{3}\right)^{d_{12}}, t \in (0, \pi/2) : \text{[9, Proposition 1.5]} \qquad (3.2.20)$$

where $d_8 = \frac{\pi^2}{24 \ln\left[\frac{1+\cosh\left(\frac{\pi}{2}\right)}{2}\right]}$, $d_9 = \frac{\pi^2}{24 \ln(3/2)}$, $d_{10} = \frac{24 \ln\left[\frac{2 \sinh\left(\frac{\pi}{2}\right)}{\pi}\right]}{\pi^2} = d_{11}$, $d_{12} = \frac{\pi^2}{24 \ln\left[\frac{2+\cosh\left(\frac{\pi}{2}\right)}{3}\right]}$ and $\{d_8\}, \{d_9, d_{10}\}, \{d_{11}, d_{12}\}$ are the sets of best possible constants for inequalities $(3.2.18), (3.2.19), (3.2.20)$ respectively, such that they hold true. Note that $(1 + \cosh t)/2$ type functions are very interesting functions as they appear often in Cusa-Huygen's type inequalities, in view of this, author [7, Theorem 2] established upper and lower bounds of this type, meanwhile they obtained another inequality [7, Lemma 1], see (3.2.21),

$$\frac{t}{\sinh t} + \cosh t > 2, t \in \mathbb{R} \setminus \{0\} \qquad (3.2.21)$$

Many researchers have developed inequalities involving mixture of trigonometric and hyperbolic functions, some of such inequalities depicted here:

$$\frac{t}{\sinh t} < \cosh t, t \in (0, \infty) : \text{[8, Lemma 2.2]} \qquad (3.2.22)$$

$$\frac{t}{\tan t} < \frac{\tanh t}{t}, t \in (0,1) : \text{[8, Remark 2.4] \& [11, Corollary 1]} \qquad (3.2.23)$$

$$\sin t \sinh t < \tan t \tanh t, t \in (0,1) : \text{[8, Corollary 2.1]} \qquad (3.2.24)$$

$$\frac{1}{\cosh t} < \frac{t^2}{\sinh^2 t} < \left(\frac{1}{\cosh t}\right)^{1/2}, t \in (0,1) : \text{[12, Theorem 2.2]} \qquad (3.2.25)$$

In the previous section, we have presented a double sided inequality (3.1.24) for trigonometric functions, now we write the double inequality [10, Theorem 2.2] for hyperbolic function, as follows:

$$2 + \cosh^2(t) > \frac{\sinh(2t)}{2t} + \frac{2 \sinh(t)}{t} > 1 + 2 \cosh(t), t \in \mathbb{R} \setminus \{0\} : \text{[10, Theorem 2.2]} \qquad (3.2.26)$$

Equivalently (3.1.26) can be written as

$$\frac{2}{\cosh(t)} + \cosh(t) > \frac{\sinh(t)}{t} + \frac{2 \tanh(t)}{t} > \frac{1}{\cosh(t)} + 2, \ t \in \mathbb{R} \setminus \{0\} : \text{[10, Remark 1]} \qquad (3.2.27)$$

Also, the inverse hyperbolic functions studied by researchers in same way as that of hyperbolic functions in view of inequalities, some of them are as follows:

$$e^{-t^2/6} < \frac{\sinh^{-1} t}{t} < e^{d_{13} t^2}, t \in (0, 1) : \text{[11, Theorem 4]} \qquad (3.2.28)$$

Due to limitations of above inequality, Bagul et al.[6] provided (3.2.29) with larger domain and proved it with the help of $\frac{t - \sqrt{1+t^2} \sinh^{-1} t}{t^2 \sinh^{-1} t}$, which negative increasing function on $(0, \infty)$,

$$e^{-t^2/6} < \frac{\sinh^{-1} t}{t} < e^{d_{13} t^2}, t \in (0, \infty) : \text{[6, Theorem 6]} \qquad (3.2.29)$$

In [18] authors gave some mixed functions inequalities, some are given in section *3.1* and *3.4*, see (3.1.33), (3.1.34), (3.4.12), (3.4.13).

3.3 $\boldsymbol{sin(t) sinh(t)}$ and $\boldsymbol{\cos(t) \cosh(t)}$ inequalities:

Recently, the sine product $\sin(t) \sinh(t)$ and cosine product $\cos(t) \cosh(t)$ have been studying extensively. In 2019 [14] Chesneau & Bagul improved the useful inequalities (3.3.1).

$$sin(t) sinh(t) \leq t^2 \text{ and } \cos(t) \cosh(t) \leq 1, t \in (0, \pi/2) \qquad (3.3.1)$$

They proposed double inequalities for functions in (3.3.1). Also, they have modified domain $(0, \pi/2)$ to $(0, \pi)$, and proved them by using infinite products and Bernoulli inequality. For $v \in (0, \pi)$, the new inequalities are given below:

$$t^2 e^{-p_1 t^4} \leq \sin t \sinh t \leq t^2 e^{-t^4/90} : \text{[14, Proposition 2.1]} \qquad (3.3.2)$$

$$e^{-p_2 t^4} \leq \cos t \cosh t \leq e^{-t^4/6}, v \in (0, \pi/2) : \text{[14, Proposition 2.3]} \qquad (3.3.3)$$

where $p_1 = -\ln(\sin v \sinh v / v^2)/v^4$ and $p_2 = -\ln(\cos v \cosh v)/v^4$. (3.3.2) and (3.3.3) provides polynomial-exponential type bounds and exponential type bounds respectively.

$$t^2\left(1-\frac{t^4}{\pi^4}\right)^{\pi^4/90} < \sin t \sinh t \ , v \in (0,\pi)\text{: [14, Proposition 2.2]} \tag{3.3.4}$$

$$\left(1-\frac{16t^4}{\pi^4}\right)^{\pi^4/96} < \cos t \cosh t \ , v \in (0,\pi/2)\text{: [14, Proposition 2.2]} \tag{3.3.5}$$

The sharpness of inequalities $(3.3.2) - (3.3.5)$ among themselves have been discussed by the authors [14].

3.4 Ratio functions inequalities:

These are one of the important functions obtained by dividing two different functions with condition that function in the denominator must be non zero. In this section, we have discussed the inequalities for ratio functions like $\frac{\sin t}{\sinh t}, \frac{\cos t}{\cosh t}, \frac{\tanh t}{\tan t}, \frac{\sin^{-1} t}{\sinh^{-1} t}, \frac{\tanh^{-1} t}{\tan^{-1} t}$. In 2022 [4] Bagul et al. and [18] Thool et al. figured out inequalities involving quotients of circular and hyperbolic functions along with their inverses. Authors in [4, 18] provided polynomial type and exponential type bounds for above mentioned ratio functions. The detail discussion on comparison of lower and upper bounds of these inequalities along with their significance is explicitly given in [4, 18] and references therein.

$$e^{-\beta_1 t^2} \le \frac{\cos t}{\cosh t} \le e^{-t^2}, a \in \left(0,\frac{\pi}{2}\right), t \in [0,a]\text{: [4, Proposition 1]} \tag{3.4.1}$$

$$e^{-\beta_2 t^2} < \frac{\sin t}{\sinh t} < e^{-t^2/3}, t \in \left(0,\frac{\pi}{2}\right)\text{: [4, Proposition 2]} \tag{3.4.2}$$

$$e^{-\beta_3 t^2} < \frac{\tanh t}{\tan t} < e^{-2t^2/3}, a \in \left(0,\frac{\pi}{2}\right), t \in (0,a]\text{: [4, Proposition 4]} \tag{3.4.3}$$

where $\beta_1 = \ln\left(\frac{\cosh a}{\cos a}\right)/a^2$, $\beta_2 = 4\ln\left(\frac{\sinh \pi}{2}\right)/\pi^2$ and $\beta_3 = \ln(\tan a / \tanh a)/a^2$. Note that, $(3.4.1) - (3.4.3)$ are providing exponential bounds. $\{\beta_1, 1\}, \{\beta_2, 1/3\}$ and $\{\beta_3, 2/3\}$ are nothing but the sets of best possible constants such that inequalities $(3.4.1) - (3.4.3)$ hold true. Recently, Bagul et al.[2] provided alternative proofs of $(3.4.1) - (3.4.2)$ using infinite products and [2, inequality 11]. As a result of $(3.4.1) - (3.4.2)$, authors obtained (3.4.4) as follows:

$$\frac{\cos t}{\cosh t} < \left(\frac{\sin t}{\sinh t}\right)^3 \text{ or } \left(\frac{\tanh t}{\tan t}\right)^{1/2} < \frac{\sin t}{\sinh t} \ , t \in (0,\pi/2)\text{: [4, Theorem 5]} \tag{3.4.4}$$

$$\frac{\sin t}{\sinh t} < \sqrt{\frac{2t+\sin 2t}{2t+\sinh 2t}} \ , t \in (0,\pi)\text{: [4, Proposition 3]} \tag{3.4.5}$$

The specialty of this inequality lies in its larger domain, where the previous inequality (3.4.2) had smaller domain. The algebraic type bounds for $\frac{\sin t}{\sinh t}$ can be seen in (3.4.6)

$$1-\frac{t^2}{3} < \frac{\sin t}{\sinh t} < 1 \ , t \in (0,\pi/2)\text{: [4, Proposition 5]} \tag{3.4.6}$$

Now, we combine Theorem 7 and Proposition 6 of [4], which results into the following inequality

$$(\cos t)^{2/3} < \frac{\sin t}{\sinh t} < \frac{2+\cos t}{2+\cosh t} \ , t \in (0,\pi/2) \tag{3.4.7}$$

$$(\cos t)^{2/3} < \frac{\sin t}{\sinh t} < (\cosh t)^{-2/3}\text{: [4, Remark 2]} \tag{3.4.8}$$

One can easily observed that, the upper bound in (3.4.7) is tighter than the upper bounds in (3.4.5), (3.4.6) and (3.4.8), where the upper bound in (3.4.2) refines upper bounds in $(3.4.5) - (3.4.8)$. The algebraic lower bound in (3.4.6) is sharper than that of in (3.4.7). Authors in [18] provided polynomial and exponential type bounds for the $\frac{\sin^{-1} t}{\sinh^{-1} t}, \frac{\tanh^{-1} t}{\tan^{-1} t}$ as follows, see (3.4.9) and (3.4.10):

$$1+\frac{t^2}{3} < e^{t^2/3} < \frac{\sin^{-1} t}{\sinh^{-1} t} < e^{\beta_4 t^2} < 1+\beta_5 t^2, t \in (0,1) \qquad (3.4.9)$$

where $\beta_4 = \frac{\pi}{\ln(3+2\sqrt{2})} - 1$ and $\beta_5 = \ln\left(\frac{\pi}{2\sinh^{-1} 1}\right)$.

$$1+\frac{2t^2}{3} < e^{2t^2/3} < \frac{\tanh^{-1} t}{\tan^{-1} t} < e^{\beta_6 t^2} < 1+\beta_7 t^2, t \in (0,b),\ b \in (0,1) \qquad (3.4.10)$$

where $\beta_6 = \frac{\tanh^{-1} b - \tan^{-1} b}{b^2 \tan^{-1} b}$ and $\beta_7 = \frac{\ln(\tanh^{-1} b/\tan^{-1} b)}{b^2}$, the sharpness among (3.4.9) and (3.4.10) can be observed analytically. Authors [18] provided some auxiliary inequalities in concern with (3.4.9) and (3.4.10), which are as follows:

$$\frac{\sin^{-1} t}{\sinh^{-1} t} < \frac{\tanh^{-1} t}{\tan^{-1} t}, t \in (0,1)\text{: [18, Corollary 1]} \qquad (3.4.11)$$

$$\frac{\sinh^{-1} t}{\sqrt{1-t^2}} > \frac{\sinh^{-1} t}{\sqrt{1+t^2}}, t \in (0,1)\text{: [18, Lemma 5]} \qquad (3.4.12)$$

$$\frac{\tanh^{-1} t}{\tan^{-1} t} < \frac{\sqrt{1+t^2}}{\sqrt{1-t^2}}, t \in (0,1)\text{: [18, Lemma 6]} \qquad (3.4.13)$$

One of such inequality [18, Lemma 7] is given by (3.1.34).

3.5 Wilker type and Cusa-Huygens type inequalities:

In sections *3.1* to *3.4,* we have discussed named inequalities like Jordan's, Kober's, Becker-Stark's, etc. and studied their layer wise evolution. In this section, we will be discussing about Wilkar type and Cusa-Huygens type inequalities. In [3], authors presented refined lower bound for hyperbolic Wilker type inequality [3, LEMMA 4] and its circular counterpart [3, REMARK 4] respectively given by (3.5.1) and (3.5.2):

$$\left(\frac{t}{\sinh t}\right)^2 + \frac{t}{\tanh t} > 2 + \frac{t(\sinh t - t)}{2(1+\cosh t)}, t \in \mathbb{R}\backslash\{0\}\text{ : [3, LEMMA 4]} \qquad (3.5.1)$$

$$\left(\frac{t}{\sin t}\right)^2 + \frac{t}{\tan t} > 2 + \frac{t(t - \sin t)}{2(1+\cos t)}, t \in \mathbb{R}\backslash\{0\}\text{ : [3, REMARK 4]} \qquad (3.5.2)$$

Note that, (3.5.1) is the sharpest Wilker type inequality of its kind on $\mathbb{R}\backslash\{0\}$. Bagul et al. [4] corroborated Wilker type inequalities for ratio functions of circular and hyperbolic functions.

$$\left(\frac{\sin t}{\sinh t}\right)^2 + \frac{\tan t}{\tanh t} > 2, t \in (0,\pi/2)\text{ : [4, Corollary 1]} \qquad (3.5.3)$$

$$\left(\frac{\sinh t}{\sin t}\right)^2 + \frac{\tanh t}{\tan t} > 2, t \in (0,\pi)\text{ : [4, Theorem 6]} \qquad (3.5.4)$$

Authors [18] have discussed about the existence and truthiness of Wilker type and Huygens type inequalities involving ratio functions like $\frac{\sin^{-1} t}{\sinh^{-1} t}, \frac{\tanh^{-1} t}{\tan^{-1} t}$ over the domain $(0,1)$. In [9] Bagul & Chesneau discussed Cusa-Huygens type inequalities, see [9, Proposition 1.3-1.5], which have already seen in sections *3.1, 3.2* of this paper.

4. Open Problems:

These are the problems stated by mathematicians, for which no one was able to provide proof till date. They arise from various situations of different discipline. Due to their usefulness, many inequalities have been established recently. These inequalities are known by various names viz., Jordan's inequality, Wilker's inequality, Huygen's inequality, Cusa-Huygen's inequality, etc. Motivated by these inequalities, we try to explore some open problems.

(I) Bagul-Chesneau's Four Inequalities:

In [10] Bagul & Cheneau established two double sided inequalities viz., (3.1.24) and its hyperbolic counterpart (3.2.26), proved using first derivative test efficiently.

One can refer their equivalent forms $(3.1.25)$ and $(3.2.27)$. Authors [10] posed two open problems, constituted by four separate inequalities, which are depicted below:

BC(i): Prove, $p + (\cos t)^p > \frac{\sin pt}{pt} + p\ \frac{\sin t}{t}\ , for\ t \in \left(0, \frac{\pi}{2}\right)\ and\ p \geq 2.$

BC(ii): Prove, $\frac{\sin qt}{qt} + q\ \frac{\sin t}{t} > 1 + q \cos t\ , for\ t \in \left(0, \frac{\pi}{2}\right)\ and\ 0 < q \leq 2.$

BC(iii): Prove, $r + (\cosh t)^r > \frac{\sinh rt}{rt} + r\ \frac{\sinh t}{t}\ , for\ t \in \mathbb{R}\backslash\{0\} and\ 0 < r \leq 2.$

BC(iv): Prove, $\frac{\sinh st}{st} + s\ \frac{\sinh t}{t} > 1 + s \cosh t\ , for\ t \in \mathbb{R}\backslash\{0\}\ and\ s \geq 2.$

In [17] Li & Guo established and proved a refined inequality of *BC(iv)* only. But, till date no one proved Bagul-Chesneau's four inequalities $i.e.$ *BC(i)-BC(iii)*, hence these inequalities attract mathematicians for further research.

(II) Since the inception of field of mathematical inequalities, researchers are more interested in studying polynomial and exponential type bounds, because of their efficient nature in terms of computation. Thool et al. pointed out [18, Lemma 7], see $(3.1.34)$. In view of this, we posed two inequalities having polynomial and exponential type bounds:

$If\ t \in (0,1),\ then\ prove\ \ \frac{t}{tan^{-1} t} + \frac{t}{tanh^{-1} t} < 2e^{-4t^4/45}\ and\ \frac{t}{tan^{-1} t} + \frac{t}{tanh^{-1} t} < 2 - \frac{8}{45}t^4.$

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